Two applications of oriented percolation

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1 Introduction

Phase transition refers to the phenomenon where a system, governed by one or a set of parameters, qualitatively changes its behavior when one or a combination of its parameters exceeds a critical value. Perhaps the most famous example is the independent bond percolation process on the d–dimensional integer lattice. It is known that, for every d, there exists a critical value $p_c = p_c(\mathbb{Z}^d)$ such that the probability of the origin percolating, $\theta(p) = \mathbb{P}_p(|C(0)| = \infty)$ is zero if the probability of edge activation $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. [8]

Another example might be the standard Galton–Watson process where the parameter of interest is the mean number of offspring, m. It is known that for $m \leq 1$ the probability of having an infinite generation tree is zero, whereas if $m > 1$, the latter probability is (strictly) positive.

In this text, we are going to look at two examples of interacting particle systems whose evolution depends on a set of parameters and can be put into correspondence with the evolution of an oriented percolation process (see the next section). The reason we are going to make this link between the two models is that the phase transition of the oriented percolation process, i.e. the fact that it percolates for some non–trivial parameter value, will be the key point in proving a desired behavior of our models, such as the indefinite survival of the particle population, for example. The percolation of the oriented percolation process will be a sort of guarantee which will imply the population survival, an event which will be of our main interest. The extinction of the population, on the other hand, is usually proved by other methods.

In the rest of this introductory section, we are going to talk about the two main tools we will be using throughout the text, namely the oriented percolation model itself and the concept of stochastic domination between measures.

1.1 Oriented percolation

Oriented percolation is a variant of the standard percolation process where each site or bond is open with a certain probability p independently of each other, but now, **bonds are oriented in a certain direction** and the fluid we supply at the origin is allowed to travel along open edges in the directions of their orientations only.

We denote by C the set of vertices that may be reached from the origin along open **directed** paths and the percolation probability is, as usual, given by $\mathbb{P}_p(|C| = \infty)$.

Like the standard percolation process, the oriented percolation also exhibits a phase transition and has proven to be very useful comparison process for interacting particle systems. The purpose of this text is to present two such examples.

In general, there can be many different models of oriented percolation depending on

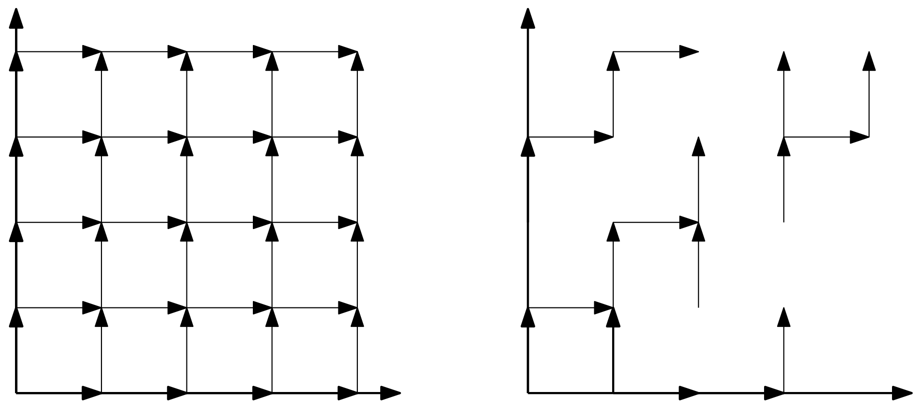

Figure 1: *On the left, an example of an oriented percolation model on $\mathbb{Z}_+ \times \mathbb{Z}_+$ where edges point on the right or upwards. On the right, a particular realization of open edges where $C = \{(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)\}$.*

our choice of the orientation of the edges. However, the process is usually considered on a space–time grid of the form $\mathbb{Z}^d \times \mathbb{Z}_+$ and the edges point in the positive time direction, so we can imagine the fluid supplied at the origin at time $n = 0$ flowing in time through the open edges.

In the remaining of this subsection, we are going to rigorously define and present the model that was studied in detail by R. Durrett in [2] and to which we will compare the contact process, our first example. It will also serve as a base for the slightly different oriented percolation model we will use for the spatial branching system on $\mathbb{Z}^d$ which we present later in the corresponding section. At the end of Section 3., we resort to Durrett's model again to obtain a result about the way the branching system on $\mathbb{Z}^d$, in the particular case of $d = 1$, evolves on the line.

1.1.1 Durrett's model

What we call Durrett's model is an oriented percolation model on a subset of $\mathbb{Z}^2$, the Durrett's integers, given by $\mathbf{L} = \{(x,n) : (x,n) \in \mathbb{Z}^2 \text{ with } x+n \text{ even}\}$, or rather its intersection with the y–positive half–plane. We imagine the sites connected with edges pointing "upwards" as illustrated on Figure 2. : every $x \in \mathbb{Z}^2$ is connected to $x + (1,1)$ and $x + (-1,1)$ by two edges. The second coordinate may be thought of as time and, in that case, we can imagine every integer $z \in \mathbb{Z}$ at time t connected to its two neighbors $z+1$ and $z-1$ at time $t+1$.

In Durrett's model, every site $x \in \mathbb{Z}^2$ is open with probability p and closed with probability $1-p$ independently of each other and when both ends of an edge are open, we can go along the edge in the direction of the orientation. More precisely, we say that y can be reached from x ($x \longrightarrow y$) if there is a sequence of neighboring open sites $x_0 = x, x_1, ..., x_m = y$. As usual, we define the cluster containing the origin by $C = \{x \in \mathbf{L} : 0 \longrightarrow x\}$ and the event of interest, the percolation itself, $\{|C| = \infty\}$.

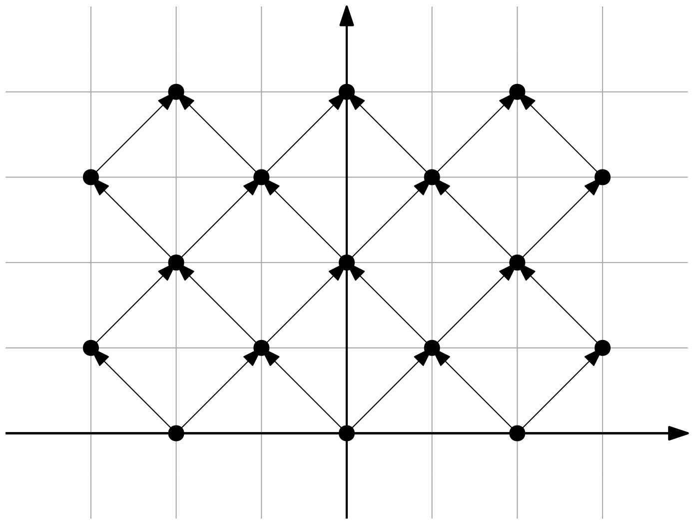

Figure 2: *Durrett's oriented percolation model*

Like the standard percolation process, Durrett's model also exhibits phase transition and we have the following theorem :

■ **Theorem 1.** *There is a value $p_c \in (0,1)$ such that if $p > p_c$, $\mathbb{P}_p(|C| = \infty) > 0$.*

The proof of Theorem 1. can be found in Durrett's original paper on oriented percolation [2]. Now, we turn to the time interpretation of the process and let $A_n^0 = \{x : (x,n) \in \mathbf{L} \text{ and } (0,0) \to (x,n)\}$ be the set of integers that can be reached in n steps through open sites starting from 0. A major point of interest in Durrett's paper is the right edge of A_n^0 (i.e. the rightmost point in A_n^0) defined by

$$r(n) = \sup A_n^0.$$

It is shown that, on the event of percolation, $r(n)$ achieves an asymptotic speed which describes the span of the infinite cluster in the space. More precisely, we have the following theorem :

■ **Theorem 2.** *$r(n)/n \to \rho \in [0,1]$ almost surely on the event of percolation.*

which says that the right edge achieves asymptotically a linear speed of ρ.

The speed constant ρ may be seen as a function of the parameter p, $\rho(p)$, with the obvious fact that $\rho(1) = 1$ since in that case $r(n) = n$ for all n. It will be important to us later to know if we can achieve every desired speed $\rho < 1$ if we make the parameter p sufficiently large. Fortunately, this is true and we also have the following theorem :

■ **Theorem 3.** *$p \mapsto \rho(p)$ is continuous for $p > p_c$. In particular, $\rho(p) \to 1$ as $p \to 1$.*

1.2 Stochastic domination

We now introduce the concept of stochastic domination of measures.

□ **Definition 1.** Let $(X, \mathcal{B})$ be a measurable space with X partially ordered and μ_1 and μ_2 two probability measures on $(X, \mathcal{B})$. We say that μ_2 **stochastically dominates** μ_1 ($\mu_1 \leq_{st} \mu_2$) if, for any bounded measurable **increasing** function f,

$$\int_X f d\mu_1 \leq \int_X f d\mu_2$$

Stochastic domination can be also defined in terms of random variables, i.e. if X and Y are two random variables taking values in $(X, \mathcal{B})$, we say that $X \leq_{st} Y$ if the law of Y stochastically dominates the law of X in the sense of the definition from above. It amounts to saying that the expectation of every bounded measurable increasing function of X is smaller than the respective expectation for Y :

$$\mathbf{E}[f(X)] \leq \mathbf{E}[f(Y)].$$

The example to keep in mind, for the purpose in our text, is $X = \{0, 1\}^S$ where S is a countable set (such as $\mathbb{Z}^d$ for example) with X itself equipped with the product topology, the corresponding Borel σ–field and partially ordered in the usual way

$$\forall \omega_1, \omega_2 \in X = \{0, 1\}^S, \quad \omega_1 \leq \omega_2 \Longleftrightarrow \omega_1(s) \leq \omega_2(s) \quad \forall s \in S.$$

As an example, suppose that $X = \{X_s : s \in S\}$ and $Y = \{Y_s : s \in S\}$ are two families of random variables taking values in $\{0, 1\}$ indexed by $S = \mathbb{Z}^d$ (i.e. X and Y are two random vectors taking values in $\{0, 1\}^{\mathbb{Z}^d}$). Suppose also that the X family of variables are i.i.d. Bernoulli variables and that we've shown that $X \leq_{st} Y$. Then, since the probability of percolation, $\mathbb{P}_p(|C| = \infty)$, is the expectation of an increasing function (namely, the indicator function of the event $\{|C| = \infty\}$), we get that the probability of Y percolating is bigger than the probability of X percolating. In particular, if the parameter p of the X family, which are i.i.d. Bernoulli, is such that $p > p_c(\mathbb{Z}^d)$, then it follows that Y percolates with positive probability. Notice that we haven't made any assumption about the mutual (in)dependence of the variables in the Y family, yet we were able to draw an important conclusion about its behavior.

The idea of stochastic domination is closely related to the one of coupling and the following theorem (Theorem 2.4 of II.2. in [5]) relates the the two concepts.

■ **Theorem 4.** *Suppose μ_1 and μ_2 are two probability measures on a measurable space $(X, \mathcal{B})$. A necessary and sufficient condition for $\mu_1 \leq_{st} \mu_2$ is that there exist a probability measure μ on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ whose marginals are μ_1 and μ_2 (i.e. μ is a coupling of μ_1 and μ_2) :*

$$\forall A \in \mathcal{B}, \quad \mu(A \times X) = \mu_1(A)$$
$$\forall A \in \mathcal{B}, \quad \mu(X \times A) = \mu_2(A)$$

and such that

$$\mu(\{(x, y) : x \leq y\}) = 1$$

i.e. μ puts all its mass over the diagonal.

The situation of the previous example is the one that we will encounter in both of our models. Namely, we are going to create a countably indexed family of Bernoulli random variables, not necessarily independent, whose percolation (in a precisely defined manner) will imply certain desired behavior of our system, such as a long term survival of a population, for example. The way we are going to prove the positive probability of percolation (and consequently the positive probability of survival) is by showing that our family of variables is **stochastically above** a family of i.i.d. Bernoulli random variables with a sufficiently high parameter p to guarantee percolation.

The most important theorem that will allow us to do this is the Liggett–Schonmann–Stacey's theorem. It says that under certain conditions of k–dependence and sufficiently large density (where by density we mean the probability of site/edge activation), the Y family will dominate an independent family having also high density.

The theorem was originally proved in [4], but here, we present the version presented in Grimmett's book on percolation theory (see [8], p.179) which is nicely stated for our needs. First, we define the notion of k–dependence for a family of random variables indexed by $S - \mathbb{Z}^d$. In the definition below, $d : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}_+$ given by $d(x, y) = \sum_{k=1}^{d} |x_k - y_k|$ is the usual distance between points in $\mathbb{Z}^d$.

□ **Definition 2.** A family of random variables $X = \{X_s : s \in \mathbb{Z}^d\}$ is called k–dependent if any two sub–families $\{X_s : s \in A\}$ and $\{X_s : s \in B\}$ are independent whenever $d(A, B) = \min\{d(x, y) : x \in A, y \in B\} > k$.

In other words, dependence can possibly exist only among variables which are at distance less than k apart. We also say that the dependence among the variables is **finite ranged**.

■ **Theorem 5.** *Let $d, k \geq 1$. There exists a non–decreasing function $\pi : [0, 1] \to [0, 1]$ satisfying $\pi(\gamma) \to 1$ as $\gamma \to 1$ such that the following holds: if $Y = \{Y_s : s \in \mathbb{Z}^d\}$ is a k–dependent family of random variables satisfying*

$$\mathbb{P}(Y_s = 1) \geq \gamma \quad \forall s \in \mathbb{Z}^d$$

then

$$Y \geq_{st} X^{\pi(\gamma)}$$

where $X^{\pi(\gamma)} = \{X_s^{\pi(\gamma)} : s \in \mathbb{Z}^d\}$ is a family of mutually independent Bernoulli random variables with mean $\pi(\gamma)$, i.e.

$$\mathbb{P}(X_s^{\pi(\gamma)} = 1) = 1 - \mathbb{P}(X^{\pi(\gamma)} = 0) = \pi(\gamma).$$

The above theorem says that whenever we deal with a family of Bernoulli random variables which is k–dependent, we can make it stochastically above an independent family of any desired density ($\pi(\gamma)$), provided that we sufficiently increase the density (γ) of our original family.

2 The contact process on $\mathbb{Z}$

2.1 Description of the system

The contact process is a model of an interacting particle system. It is a continuous time Markov process with state space $\{0,1\}^S$ where S is a finite or countable graph. In the rest of this section, we are going to suppose that $S = \mathbb{Z}$. The process is usually interpreted as a model for the spread of an infection : if the state of the process at a given time is $\xi : \mathbb{Z} \to \{0,1\}$, then an individual at position x in $\mathbb{Z}$ is **infected** if $\xi(x) = 1$ and **healthy** if $\xi(x) = 0$. We can also imagine the state as a subset of $\mathbb{Z}$ containing the positions of the infected individuals, i.e. $x \in \xi \Leftrightarrow \xi(x) = 1$, with slight abuse of notation.

Infected individuals become healthy at a constant rate equal to 1, while healthy ones become infected at a rate proportional to the number infected neighbors ($\lambda \times$ number of infected neighbors). More precisely, if $x \in \xi_t$, then

$$\mathbb{P}(x \notin \xi_{t+h} | \xi_t) = h + o(h) \tag{1}$$

and if $x \notin \xi_t$, then

$$\mathbb{P}(x \in \xi_{t+h} | \xi_t) = \lambda(\xi_t(x-1) + \xi_t(x+1))h + o(h) \tag{2}$$

Following the example in [3], we are going to slightly modify this standard definition of the contact process, setting $\lambda = 1$ and letting the healing rate to be δ (instead of 1). The qualitative behavior of the process depends on the ratio between the two rates, so we are not making a fundamental change to the model. The dynamics that are going to govern our modified process are thus the following :

$$\mathbb{P}(x \notin \xi_{t+h} | \xi_t) = \delta h + o(h) \text{ if } x \in \xi_t \tag{3}$$

$$\mathbb{P}(x \in \xi_{t+h} | \xi_t) = (\xi_t(x-1) + \xi_t(x+1))h + o(h) \text{ if } x \notin \xi_t \tag{4}$$

2.2 Preliminaries and statement of the main theorem

We will prove that, depending on the value of δ, the infection either disappears with probability one or spreads among individuals indefinitely with positive probability. This behavior should be intuitively clear : if δ is very big, the healing will be much faster than the transmission of the infection among neighbors and the latter will die out at some point in time, and, vice–versa, if δ is very small (imagine $\delta = 0$), then the infection has a chance on survival among the population.

We will denote by ξ_t^A the state of the system at time t starting with $\xi_0^A = A$ and will write ξ_t^0 for $\xi_t^{\{0\}}$. We also define the critical value for δ as

$$\delta_c = \sup\{\delta : \mathbb{P}(\xi_t^0 \neq \emptyset \text{ for all } t) > 0\} \tag{5}$$

and our goal will be to prove the following theorem :

■ **Theorem 6.** *The critical value for the contact process on $\mathbb{Z}$ satisfies* $0 < \delta_c \leq 2$.

We will prove the difficult part that $\delta_c > 0$ which means that for all sufficiently small values od δ, the probability that the infection disappears is non–zero.

There are two main ideas in the proof :

- using appropriate space–time normalization, compare the model, with $\delta = 0$, to an oriented percolation process in such a way that the percolation of the latter will imply the indefinite survival of the infection
- extend the previous result, by continuity, for small values of δ

The first point alone proves a trivial result — namely, that the infection doesn't die out in the absence of healing. However, its importance lies in the fact that it allows us to prove the main result — by increasing a bit the parameter δ, we will only decrease the probability parameter p of the already established oriented percolation process. If we cared to choose it sufficiently large in the first place, it will still be enough to guarantee percolation.

2.3 Note on the process construction

The contact process can be constructed using two families of independent Poisson processes defined on a probability space. The first one, $\mathcal{H} = \{H_x : x \in \mathbb{Z}\}$, indexed by $\mathbb{Z}$ contains processes with rate δ and the second one, $\mathcal{I} = \{I_{x,y} : x, y \in \mathbb{Z}, |x - y| = 1\}$, contains processes with rate 1 and is doubly–indexed by the neighbors in $\mathbb{Z}$. If $s \in H_x$, then the individual at position x is healed at time s if infected and, similarly, if $s \in I_{x,y}$, the infection is transmitted from the individual at position x to its neighbor at y at time s if the former carried the infection the moment before s. The reader is referred to [11] (Chapter 6., Section 6.2.) for a complete and clear explanation.

2.4 Proof of the main theorem

Proof. Suppose that $\delta = 0$ and that $\xi_0 = A_0 = [-L, L]$. Since an individual cannot heal once it has been infected, the infection will propagate in a flood fashion, with the state at every time being an interval of the form $\xi_t^{[-L,L]} = [l_t, r_t]$ (in what follows, we will omit the superscript $[-L, L]$).

The end points of the interval of infected individuals will move respectively to $l_t - 1$ and $r_t + 1$ at rate equal to 1. It means that the right end point (the same holds analogously for the left one) does a one–direction random walk, jumping by one distance unit, i.e. infecting one individual, to the right every $\mathsf{Exp}(1)$ time units. Consequently, the position of the right end point of the interval at time t, r_t, will have the same distribution as $L + \mathbf{N}(t)$ where $\mathbf{N}(t)$ is a standard Poisson process.

Denote by A_m the interval $[-L, L]$ centered at $2mL$, i.e. $A_m = 2mL + A_0 = 2mL + [-L, L] = [(2m - 1)L, (2m + 1)L]$.

Before we proceed, we briefly explain the space–time normalization and the connection we will create with the oriented percolation process. The time–normalization refers to the fact that we will "discretize" the process by looking its evolution on specific time instants of the form nT where T will be suitably chosen. Let $\mathbf{X} = \{X_{m,n} : (m,n) \in \mathbb{Z} \times \mathbb{Z}_+ \text{ with } m+n \text{ even}\}$ be a family of Bernoulli random variables indexed by the Durrett integers. We let $X_{m,n} = 1$ if the following two events are realized

- the process starting with the individuals in $A_m = [(2m-1)L, (2m+1)L]$ infected at time nT, has the individuals in the two neighboring intervals of length $2L$, A_{m-1} and A_{m+1}, also infected by the time $(n+1)T$.
- during the whole time span $[nT, (n+1)T)$, none of the individuals beyond, i.e. on the right of $(2m+2)L$ (and respectively on the left of $(2m-2)L$) have been infected

The first assumption relates the percolation of the associated oriented percolation process to the random variables in $\mathbf{X}$ to the indefinite propagation of the infection. Indeed, suppose that $\mathbf{X}$ percolates, i.e. that there is, with positive probability, an infinite cluster arising at the origin. It would mean that at every $t = nT$, there is at least one contiguous block of $2L$ infected individuals somewhere in the space.

The second assumption allows us to control the dependence of the random variables $X_{m,n}$ which are **not** independent in the way they are defined. Two variables $X_{m,n}$ and $X_{k,l}$ will be independent only if the Durrett distance between them is strictly greater than 1 which makes the $\mathbf{X}$ family 1–dependent.

It is useful to note, as it has been a source of confusion in some of the literature, that the variables in $\mathbf{X}$, by the way we construct the contact process, are measurable with respect to the "endings" of some sub–family of of Poisson processes and not the contact process ξ itself.

In what follows, we will restrict our attention to the case $(m,n) = (0,0)$ and we will show that if the parameter $L \geq L_\varepsilon$, then $\mathbb{P}(X_{0,0} = 1) \geq 1 - \varepsilon$. Keep in mind that we always suppose $\delta = 0$ and that we start the process with $\xi_0 = A_0 = [-L, L]$. In view of the idea discussed earlier, we would like to have the individuals in $A_1 = [L, 3L]$ and $A_{-1} = [-3L, -L]$ infected at time T, but those at the left of $-4L$ and on the right of $4L$ healthy. So, in order to make $\mathbb{P}(X_{0,0} = 1)$ close to one, we need to make sure the infection localizes somewhere between $[-4L, 4L]$ and $[-3L, 3L]$ with high probability.

By the large deviation result on the Poisson process (Theorem 16.), if we choose L very large, the probability that the infection after time $T = 2.5L$ has spread within a small factor γ around $L + 2.5L = 3.5L$ will be very large as well. In particular, after $2.5L$ time units, the infection will have spread to an individual located between $3L$ and $4L$ with very high probability.

$$
\begin{aligned}
\mathbb{P}(3L \le r_T \le 4L) = \\
\mathbb{P}(3L \le L + \mathbf{N}(2.5L) \le 4L) = \\
\mathbb{P}(2L \le \mathbf{N}(2.5L) \le 3L) = \\
\mathbb{P}(-0.5L \le \mathbf{N}(2.5L) - 2.5L \le 0.5L) = \\
\mathbb{P}(-0.2 \le \frac{\mathbf{N}(2.5L)}{2.5L} - 1 \le 0.2) = \\
\mathbb{P}\left(\left|\frac{\mathbf{N}(2.5L)}{2.5L} - 1\right| \le 0.2\right) \ge \\
1 - \exp(-CL)
\end{aligned}
\tag{6}
$$

The same reasoning applies for the position of the left end point at time, l_T. Choosing a large enough L, at time T, with high probability, all the individuals in A_1 and A_{-1} will be infected and those beyond distance $4L$ of the origin will not be infected.

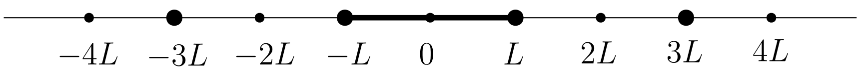

Figure 3: The infection at time $t = 0$

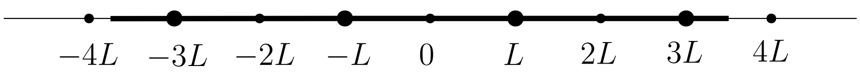

Figure 4: *The infection at time $t = T = 2.5L$. If L is chosen large enough, the typical number of newly infected individuals on the right (and on the left) will be around $2.5L$. The choice of $2.5\times$ is somewhat arbitrary — the important thing is to infect at least $2L$ individuals, i.e. the neighboring block, but not too much.*

Now, we want to show that the infection still propagates indefinitely with positive probability even if the healing rate is non–zero. We adopt the same idea which is to show that $\mathbb{P}(X_{0,0} = 1)$ can be made arbitrarily large.

We already know that for every $\varepsilon > 0$ and $\delta = 0$, $\mathbb{P}(X_{0,0} = 1)$ can be made greater than $1 - \varepsilon$ by choosing a large enough L. Keeping the same L, we can make $\mathbb{P}(X_{0,0} = 1)$ greater than, say, $1 - 2\varepsilon$ (that is, still arbitrarily close to 1) for a non–zero $\delta < \delta_\varepsilon$. The way to do this is rather crude and consists of asking for no healing among the individuals in $[-4L, 4L]$ during $[0, T]$. More precisely, we let

$$\mathcal{D} = \{(x, H_n^x) : n \ge 1, x \in \mathbb{Z}\} \tag{7}$$

be the set of healing moments and define the event

$$A = \{\mathcal{D} \cap [-4L, 4L] \times [0, T] = \emptyset\}. \tag{8}$$

The occurrence of A, in addition to the two previous events which make $X_{0,0} = 1$, will guarantee that the infection spreads as before to the two neighboring blocks even in the presence of healing. Given the construction of the contact process using the Poisson processes, A will occur if there are no “arrivals” in a finite interval of time for a finite number of such processes (those associated to the individuals in $[-4L, 4L]$).

The probability of A’s occurence, $\mathbb{P}(A) = \exp(-\delta \times T \times (8L+1))$ and can be made arbitrarily large ($\geq 1-\varepsilon$) by choosing a sufficiently small δ_ε. The fact that A is independent of the previous two events (they involve the two different families of Poisson processes) and the probabilities of all three can be made arbitrarily large, allows us to make $\mathbb{P}(X_{0,0} = 1)$, which is the the probability of their intersection, also, arbitrarily close to 1.

$\square$

The argument of how everything works, the connection with the oriented percolation and the handling of the 1–dependence, we did them first in detail for the spatial branching system of the next section. As they are very similar and can be applied here almost unchanged, we refer the reader to the end of that section for a more clear treatment of the proof.

3 Spatial branching system with local regulation

3.1 Description of the system

The second example we are going to look at is a model invented by M. Birkner & A. Depperschmidt and presented in [6] in which particles live on discrete locations indexed by $\mathbb{Z}^d$. On every site, there can be an arbitrary number of particles and the state of the system at time t is given by a function $\xi_t : \mathbb{Z}^d \to \mathbb{N}$. Unlike the contact process, the spatial branching system is going to be a **discrete–time** model. We will refer to time intervals as **epochs**.

At every epoch, particles will first reproduce on every site and then the created offspring displace themselves on the neighboring sites by moving around. The reproduction will depend on two factors

- the intrinsic reproduction rate which says that an individual particle has on average $m > 1$ offspring in the absence of competition
- the competition with the neighboring particles which says that every individual at position y reduces the reproductive success of an individual at position x by an amount $\lambda_{xy} \geq 0$.

More precisely, an individual at position $x \in \mathbb{Z}^d$ in the n–th epoch will have a **Poisson random number** of offspring with mean given by

$$\left(m - \sum_{y \in \mathbb{Z}^d} \lambda_{xy} \xi_n(y) \right)^+ \tag{9}$$

where $\xi_n(y)$ denotes the number of individuals at position $y \in \mathbb{Z}^d$ in the n–th epoch.

Once created, offspring take an independent random walk step according to a kernel p. We now state the formal description of the model [6].

We assume that the transition kernel $\mathbf{p} = (p_{x,y})_{x,y \in \mathbb{Z}^d}$ and the competition kernel $\lambda = (\lambda_{x,y})_{x,y \in \mathbb{Z}^d}$ satisfy the following conditions :

(A1) : The kernel $\mathbf{p} = (p_{x,y})_{x,y \in \mathbb{Z}^d} = (p_{0,y-x})_{x,y \in \mathbb{Z}^d}$ is a zero mean aperiodic stochastic kernel with finite range $R_p \geq 1$, that is, for all $x, y \in \mathbb{Z}^d$, $p_{xy} = 0$ if $\|x - y\|_\infty > R_p$.

(A2) : The kernel $\lambda = (\lambda_{xy})_{x,y \in \mathbb{Z}^d} = (\lambda_{y-x})_{x,y \in \mathbb{Z}^d}$ is also with finite range $R_\lambda \geq 1$ and the self–competition term $\lambda_0 := \lambda_{00} > 0$ is strictly positive.

For a configuration $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, we define the **expected number** of offspring generated at site x by

$$f(\eta, x) := \eta(x) \left(m - \lambda_0 \eta(x) - \sum_{y \neq x} \lambda_{xy} \eta(y) \right)^+ \tag{10}$$

and the **expected number** of individuals at x in the next generation if the present configuration is η by

$$F(\eta, x) := \sum_{y \in \mathbb{Z}^d} f(\eta, y) p_{yx} \tag{11}$$

Now, given ξ_n, the configuration at the n–th epoch, the configuration at the next epoch arises as

$$\xi_{n+1}(x) = \mathbf{N}^{(x,n)}(F(\xi_n, x)) \tag{12}$$

where $\{\mathbf{N}^{(x,n)} : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+\}$ is a family of independent standard Poisson processes on $\mathbb{R}_+$, i.e. $\mathbf{N}(t) \sim \text{Poisson}(t)$.

The main question which we address is the long–term survival of the population.

3.2 The (modified) oriented percolation model

We will tie the spatial branching system to an oriented percolation model which is slightly different from Durrett's model described earlier. Since we are interested in the evolution in time of a population living on $\mathbb{Z}^d$, the underlying graph for the oriented percolation model will be $G = \mathbb{Z}^d \times \mathbb{Z}_+ = \{(x,n) : x \in \mathbb{Z}^d, n \in \mathbb{Z}_+\}$ with the set of oriented edges being $E = \{(x,n) \to (y, n+1) : n \in \mathbb{Z}_+, \|x - y\|_\infty = 1\}$. Notice that there are more edges than in the usual $\mathbb{Z}^{d+1}$ cut in half : from every site x, there are 3^d edges connecting it to each one of his neighbors in the next generation.

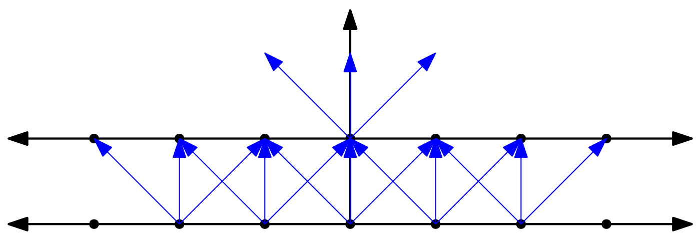

Figure 5: *Portion of the graph corresponding to the oriented percolation model on $\mathbb{Z} \times \mathbb{Z}_+$. Every site is connected to each of its three $\|\cdot\|_\infty$–neighbors.*

We designate each **site** as open with probability p and closed with probability $1-p$ independently of each other. If $k \leq l$, we say there is an **open path** from (x_k, k) to (x_l, l) (we denote $(x_k, k) \to (x_l, l)$) if there is a sequence of open sites $(x_k, k), (x_{k+1}, k+1), ..., (x_l, l)$ such that $\|x_{i+1} - x_i\|_\infty = 1$ for $i = k, ..., l-1$. Similarly, we let $A_n^0 = \{x \in \mathbb{Z}^d : (0,0) \to (x,n)\}$ be the set of sites at time n that can be reached from the origin and the define the critical parameter, as usual, by $p_c = \inf\{p : \mathbb{P}(A_n^0 \neq \emptyset, \forall n) > 0\}$.

Durrett's oriented percolation model can be **embedded** in this new oriented percolation model in an obvious way — this means that the latter exhibits a phase transition phenomenon as well and percolates for some non–trivial value of the parameter p.

3.3 Indefinite survival

3.3.1 Preliminaries and statement of the main theorem

The maximal mean number of particles that can be present at a site at any given moment is $m^*_{\lambda_0} = m^2/(4\lambda_0)$ and if there are more than $M_{\lambda_0} = m/\lambda_0$ particles at some site, no offspring is produced. We also denote by $\kappa = \sum_{x \neq 0} \lambda_{0x}$ the total "non–diagonal" competition. All of these quantities will be used in the proof of the following main theorem (the first two are illustrated on Figure 9.) :

■ **Theorem 7.** *For each $m \in (1, 4)$ and p satisfying $(A1)$, there are choices of positive numbers $\lambda^*_0 = \lambda^*_0(m, p)$ and $\kappa^* = \kappa^*(m, p)$ such that if $\lambda_0 \leq \lambda^*_0$ and $\kappa = \sum_{x \neq 0} \lambda_{0x} \leq \kappa^* \lambda_0$, then, the population survives with positive probability, that is,*

$$\mathbb{P}_{\xi_0}[\forall n \in \mathbb{N}, \exists x \in \mathbb{Z}^d \text{ such that } \xi_n(x) > 0] > 0 \tag{13}$$

for all ξ_0 with $f(\xi_0, x) > 0$ for some $x \in \mathbb{Z}^d$.

To prove Theorem 7., we are going to further "discretize" the process by considering its evolution on a sub–grid of the form $\mathbb{Z}^d \times n^*\mathbb{Z}_+$ where n^* is carefully chosen so that the evolution process restricted at these specific times of the form kn^* can be compared to (finite–ranged) dependent oriented percolation on the same sub–grid.

We begin by introducing some necessary ingredients and definitions for the proof which is somewhat technically involved.

At first, we fix an $\widetilde{m} \in (1, m)$ and define n^* as the smallest integer n such that $p^n_{0x}\widetilde{m}^n \geq 1$ for all x with $\|x\|_\infty \leq 1$, i.e.

$$n^* = \min\left\{n \in \mathbb{N} : p^n_{0x}\widetilde{m}^n \geq 1 \quad \forall x \text{ such that } \|x\|_\infty \leq 1\right\} \tag{14}$$

By the Local Central Limit Theorem (see Theorem 14. in the respective section in the Appendix), n^* is **finite** and well–defined.

We also define the set $\mathbf{J} = \left\{(x, n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : 0 \leq n \leq n^*, p^n_{0x} > 0\right\}$ — those are the sites to which transition is possible starting from $(x, n) = (0, 0)$.

The $\widetilde{m}$ we've chosen is a multiplication factor that we will be able to guarantee at every time–space point in $\mathbf{J}$. Notice that, due to the competition, particles cannot reproduce with a rate m, but we can make them reproduce with a rate $\widetilde{m} < m$ by sufficiently reducing the competition. Starting from $x = 0$ at time $n = 0$, we will arrange that particles be multiplied by at least $\widetilde{m}$ in **every epoch** and on **every site** by carefully choosing ε_1, ε_2 and sufficiently reducing the competition by choosing small λ^*_0 and κ^*.

To illustrate the idea, suppose there are more than, say, R particles at $\xi_0(0)$. Then, there will be more than $p_{0x}^n \widetilde{m}^n R$ particles at $\xi_n(x)$ for $(x,n) \in \mathbf{J}$, i.e. more than R particles at $\xi_{n^*}(x)$ for those x with $\|x\|_\infty \leq 1$. The reason n^* is finite is because, due to the finite range and aperiodicity of the transition kernel p, particles will multiply at a faster rate than they are capable of spreading themselves in the space and after a certain amount of time the geometric multiplication will compensate for the spreading in any finite neighborhood of $x = 0$.

3.3.2 A result about the deterministic system

In this section, we are going to consider the deterministic system

$$\zeta_{n+1}(x) = F(\zeta_n, x) + \delta_n(x) \tag{15}$$

where $F(\zeta_n, x)$ is as before and $\delta_n(x)$ is a perturbation at time n at the site x that is supposed to satisfy $\delta_n(x) \geq -F(\zeta_n, x)$. In this deterministic variant, we replaced the Poisson random variables with their means, $F(\zeta_n, x)$, and introduced a perturbation term.

Keep in mind throughout this section that the perturbation we will later define will be $\delta_n(x) = \mathbf{N}^{(x,n)}(F(\xi_n, x)) - F(\xi_n, x)$ which automatically transforms the deterministic system (ζ_n) into our initial stochastic system (ξ_n) :

$$\begin{aligned} \xi_{n+1}(x) &= F(\xi_n, x) + \delta_n(x) \\ &= F(\xi_n, x) + \mathbf{N}^{(x,n)}(F(\xi_n, x)) - F(\xi_n, x) \\ &= \mathbf{N}^{(x,n)}(F(\xi_n, x)) \end{aligned} \tag{16}$$

In the rest of this section, the perturbation is assumed to be unknown. We will see, however, that imposing certain conditions on it will allow us to arrive at a desired result.

Namely, the main result about the deterministic system, stated in Lemma 8., will be the proof that, under suitable assumptions on the perturbation term, an occupied site (see Definition 3.) at time $n = 0$ (or $n = k$) leads to its neighbors being also occupied at time $n = n^*$ (or $n = k + n^*$).

□ **Definition 3.** Let $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$. For a pair of positive numbers $(\varepsilon_1, \varepsilon_2)$, we will say that a site x is $(\varepsilon_1, \varepsilon_2)$-occupied (or just **occupied**) with respect to η if

$$\begin{aligned} &(1) : \eta(x) \in [\varepsilon_1 M_{\lambda_0}, (1-\varepsilon_2) M_{\lambda_0}] \\ &(2) : \eta(y) \leq (1-\varepsilon_2) M_{\lambda_0} \text{ for all } y \text{ such that } \|x-y\|_\infty \leq R_\lambda \end{aligned} \tag{17}$$

To quote [6], a site is occupied if “there are enough particles there, and not too many in the neighborhood”.

For the moment, we should think of the numbers ε_1 and ε_2 as two parameters that we will tailor later in such a way that everything will work just great. An occupied site is, in particular, a site that has non–zero number of particles and we will prove the main theorem of this section by showing that **an infinite chain of occupied sites appears with positive probability in epochs which are multiples of n^*.**

We consider the following two assumptions for the perturbation term :

$(B1)_{\varepsilon_2}$: $F(\zeta_n, x) + \delta_n(x) \leq (1-\varepsilon_2)M_{\lambda_0}$ for all $(x, n) \in \mathbf{X}$
$(B2)_{\delta,K}$: $[F(\zeta_n, x) \geq K \implies |\delta_n(x)| \leq \delta F(x, \zeta_n)]$ for all $(x, n) \in \mathbf{X}$

where $\mathbf{X} = \left\{(y, n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : 0 \leq n < n^*, \|y\|_\infty \leq n(R_p + R_\lambda)\right\}$.

The first one simply says that the number of particles is kept uniformly bounded through all epochs $1 \leq n \leq n^*$ and on all sites that may somehow be concerned by the evolution of the particles at $\zeta_0(0)$.

The second one says that if the number of particles on a particular site is sufficiently high, then the relative deviation of the perturbation with respect to that current number of particles will be small. It means that the number of particles on the same site in the next epoch will be within a small factor of the current number of particles on the site : $\zeta_{n+1}(x) = F(\zeta_n, x) + \delta_n(x) \geq (1-\delta)F(\zeta_n, x)$ if $F(\zeta_n, x) \geq K$.

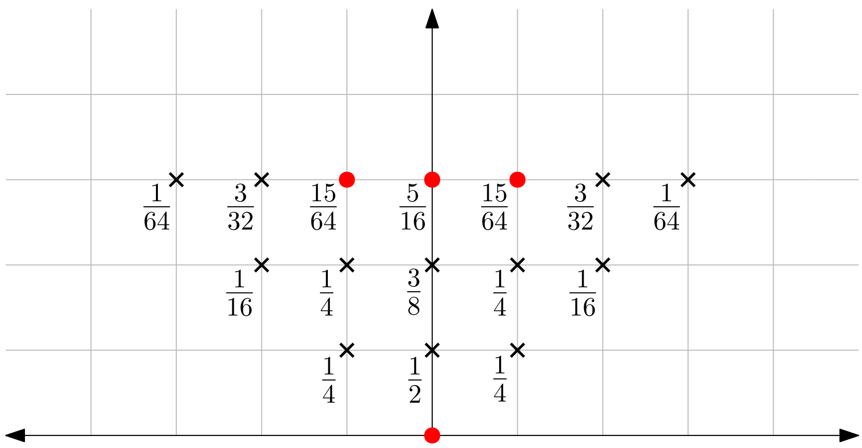

Figure 6: *Sketch of the points of the set* $\mathbf{J}$ *when the time–space evolution grid is* $\mathbb{Z}\times\mathbb{Z}_+$, $\widetilde{m} = 1.7$ *and the transition kernel is given by* $p_{+1} = p_{-1} = 1/4$ *and* $p_0 = 1/2$. *At time* $n = n^* = 3$, *we already have* $\widetilde{m}^{n^*} p_{0x}^{n^*} \geq 1$ *for all* $\|x\|_\infty \leq 1$.

Notice that, given the stochastic perturbation we will later define, that is $\delta_n(x) = \mathbf{N}^{(x,n)}(F(\zeta_n, x)) - F(\zeta_n, x)$, whether or not $(B1)_{\varepsilon_2}$ and $(B2)_{\delta,K}$ hold will depend on the random outcome $\omega \in \Omega$ and we could speak of the **probability** of either one of the assumptions being valid.

We state now the important result for the deterministic system.

■ **Lemma 8.** *Assume that m and p are as before. For each $K > 0$ and δ satisfying $m(1-\delta) > \widetilde{m} > 1$, there are choices of positive numbers $\varepsilon_1, \varepsilon_2, \lambda_0^*$ and κ^* such that whenever $\lambda_0 \le \lambda_0^*$ and $\kappa \le \kappa^* \lambda_0$, the following holds :*

$$\zeta_0(0) \text{ is } (\varepsilon_1, \varepsilon_2)\text{-occupied}, (B1)_{\varepsilon_2}, (B2)_{\delta,K} \text{ are satisfied } \implies$$
$$\zeta_{n^*}(x) \text{ are } (\varepsilon_1, \varepsilon_2)\text{-occupied for all } x \text{ with } \|x\|_\infty \le 1.$$

To prove that there is a positive probability of an infinite chain of occupied sites, and thus positive probability of an indefinite survival of the population, we will show that sites where both $(B1)_{\varepsilon_2}$ and $(B2)_{\delta,K}$ are satisfied percolate on the subgrid $\mathbb{Z}^d \times n^*\mathbb{Z}_+$, i.e., there exists, with positive probability, at least one infinite chain of sites $\mathbf{C} = ((x_0, 0), (x_1, n^*), (x_2, 2n^*), (x_3, 3n^*), ...)$ with $\|x_{k-1} - x_k\|_\infty \le 1$ such that $(B1)_{\varepsilon_2}$ and $(B2)_{\delta,K}$ are true on every (x_k, kn^*) when the latter is viewed as the origin.

To be able to establish the percolation, we would need to make sure that $(B1)_{\varepsilon_2}$ and $(B2)_{\delta,K}$ both occur with sufficiently high probability. For instance, the fact that the above lemma is valid for every K is crucial because the probability of $(B2)_{\delta,K}$ happening can be made as high as desired only if we are allowed to choose K as large as needed.

We will postpone the proof of the lemma to a later section and will show how it is used to prove the main theorem.

3.3.3 Proof of the main theorem

We may look at sites where $(B1)_{\varepsilon_2}$ and $(B2)_{\delta,K}$ hold as **fecund** sites which favor reproduction. An occupied site at time $n = 0$ (with number or particles $\in [\varepsilon_1 M_{\lambda_0}, (1-\varepsilon_2)M_{\lambda_0}]$) which is at the same time a fecund site, leads to his neighbors being occupied at time $n = n^*$. Now, if any of those sites happens to be fecund, too, its neighbors will be occupied at time $n = 2n^*$, and so on. That being said, an $(\varepsilon_1, \varepsilon_2)$–occupied site at time $n = 0$ and a percolating time–oriented cluster of fecund sites starting from this site are sufficient for long–term survival. See Figure 7. for an example of survival scenario.

The remaining goal to finish the proof of Theorem 7. is thus to establish the percolation of fecund sites.

We define the site–events :

$$\mathbf{A}(x,n) = \left\{\mathbf{N}^{(y,k)}(m_{\lambda_0}^*) \le (1-\varepsilon_2)M_{\lambda_0}, \forall (y,k) \in (x,n) + \mathbf{X}\right\}$$
$$\mathbf{B}(x,n) = \left\{\sup_{t \ge K}\left|\frac{\mathbf{N}^{(y,k)}(t)}{t} - 1\right| \le \delta, \forall (y,k) \in (x,n) + \mathbf{X}\right\}$$

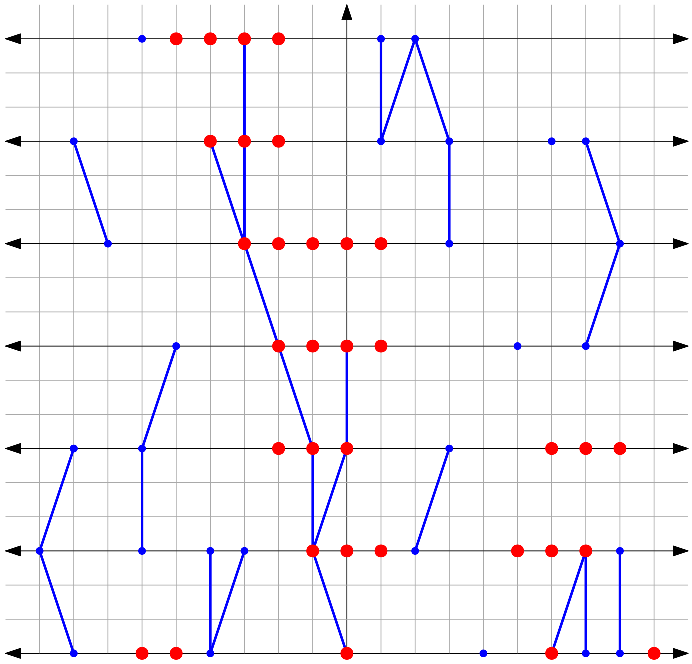

Figure 7: *Percolation on the grid* $\mathbb{Z} \times 3\mathbb{Z}_+$. *Good sites are shown with small blue dots and connected with blue lines whenever they are adjacent to each other. Occupied sites are shown with larger red dots. At time* $n = 0$, *there are five occupied sites, two of which are also fecund and lead to their neighbors being also occupied at time* $n^* = 3$. *Ultimately, only one of them* $(x = 0)$ *succeeds in spreading its occupancy thanks to the percolating cluster of good sites arising at* $(x, n) = (0, 0)$.

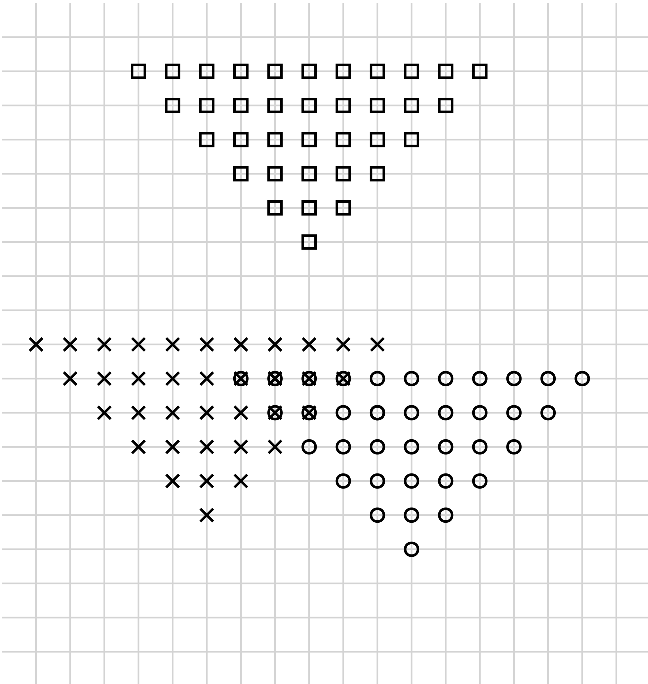

Figure 8: *The Poisson processes* $\mathbf{F}(x,n)$ *arising from three arbitrary different sites. Obviously, the events* $\{\mathbf{F}(\circ)$ *is fecund*$\}$ *and* $\{\mathbf{F}(\times)$ *is fecund*$\}$ *are not independent since they involve overlapping Poisson processes. However,* $\mathbf{F}(\square)$ *is independent from the two others because the time distance to each of them is greater than* n^**. Notice also that spatial distance greater than* $2n(R_p + R_\lambda)$ *would also imply independence.*

Notice that $m^*_{\lambda_0}$ is the maximum possible number of particles at any site, so $\mathbf{A}(x,n)$ implies the assumption $(B1)_{\varepsilon_2}$ on (x,n). Similarly, $\mathbf{B}(x,n)$ implies $(B2)_{\delta,K}$ on (x,n). Consequently, we call a site (x,n) **fecund** if both $\mathbf{A}(x,n)$ and $\mathbf{B}(x,n)$ hold (such sites are called **good** in the original article [6]). Formally, we group the concerned Poisson processes arising from (x,n) in a set

$$\mathbf{F}(x,n) = \left\{\mathbf{N}^{(y,k)} : (y,k) \in (x,n) + \mathbf{X}\right\}$$

and say that $\mathbf{F}(x,n)$ **is fecund** if $\mathbf{A}(x,n) \cap \mathbf{B}(x,n)$ holds. By our previous discussion, we need to establish the percolation of fecund sites as it provides us with an infinite chain $\mathbf{C} = ((x_0,0),(x_1,n^*),(x_2,2n^*),(x_3,3n^*),...)$ which we may "root" at an occupied site at time $n = 0$ and obtain an infinite chain of occupied sites leading to indefinite survival of our population.

We will be analyzing the random field

$$\left(\mathbb{1}_{\{\mathbf{F}(x,n) \text{ is fecund}\}}\right)_{(x,n)\in\mathbb{Z}^d\times n^*\mathbb{Z}_+} \tag{18}$$

which consists of Bernoulli random variables (one for each site (x,n)) with parameter $p = \mathbb{P}\big(\mathbf{F}(x,n) \text{ is fecund}\big)$ and which corresponds to the oriented percolation model i.e.

two sites (x, n) and (y, k) are adjacent to each other iff $|n - k| = n^*$ and $\|x - y\|_\infty \leq 1$.

We know, by results on independent site oriented percolation, that time–oriented percolation does occur with positive probability for a sufficiently high probability of site–activation p. It seems all we have to do is to take care of the probability $p = \mathbb{P}\big(\mathbf{F}(x, n) \text{ is fecund}\big)$. There is a small catch, however : our sites are, again, as in the example with the contact process, **not** being fecund independently of each other (see Figure 8.), i.e. $\mathbf{F}(x, n)$ and $\mathbf{F}(y, k)$ are **not** independent if $\|x - y\|_\infty \leq 2n(R_p + R_\lambda)$ (or if $|n - k| \leq n^*$, but this is of no concern because we restricted ourselves on sites which are already at time distance n^* from each other). Fortunately, since the dependence among sites is finite–ranged, Liggett–Schonmann–Stacey's theorem (see Theorem 5. in the Introduction) allows us to easily remedy this problem by just increasing the probability of site–activation.

More precisely, there is a value p_0 given by the results on independent site oriented percolation for which percolation occurs with non–zero probability if the underlying random field consists of **independent** Bernoulli random variables. However, the random variables in our field given in (18) are not independent, but rather $k-$dependent. For the value of p_0 from above, Liggett–Schonmann–Stacey's theorem gives us another value p_1 such that if $\mathbb{P}\big(\mathbf{F}(x, n) \text{ is fecund}\big) \geq p_1$, the field in (18), although not consisting of entirely independent variables, will nevertheless be stochastically above a random field of independent Bernoulli variables with parameter p_0 and will thus also percolate with non–zero probability.

At this point we should care to be more precise, since Liggett–Schonmann–Stacey's theorem gives the result about a random field indexed only by $\mathbb{Z}^d$. If $\mathbb{P}\big(\mathbf{F}(x, n) \text{ is fecund}\big) \geq p_1$, then the law of the field $\big(\mathbb{1}_{\{\mathbf{F}(x,k) \text{ is fecund}\}}\big)_{x \in \mathbb{Z}^d}$, indexed by $\mathbb{Z}^d$ with $n = k$ fixed, which we denote by $\nu(k)$, will dominate the product measure $\mu(k) = \bigotimes_{\mathbb{Z}^d} \text{Bern}(p_0)$. Finally, the law of our starting field (18) is the product measure $\bigotimes_{k \in n^* \mathbb{Z}_+} \nu(k)$ (since the random variables are time–independent) which dominates $\bigotimes_{k \in n^* \mathbb{Z}_+} \mu(k) = \bigotimes_{\mathbb{Z}^d \times n^* \mathbb{Z}_+} \text{Bern}(p_0)$.

Let us finally show now that the probability $p = \mathbb{P}\big(\mathbf{F}(x, n) \text{ is fecund}\big)$ can be made as high as desired. Since $\{\mathbf{F}(x, n) \text{ is fecund}\} = \mathbf{A}(x, n) \cap \mathbf{B}(x, n)$, it suffices to show that both $\mathbf{A}(x, n)$ and $\mathbf{B}(x, n)$ can be realized with high probability. By translation invariance, we will continue working with the site $(x, n) = (0, 0)$.

$$\begin{aligned} \mathbb{P}\big(\mathbf{X}(0,0) \text{ is good}\big) &= \mathbb{P}\big(\mathbf{A}(0,0) \cap \mathbf{B}(0,0)\big) \\ &= \mathbb{P}\big(\mathbf{A}(0,0)\big) + \mathbb{P}\big(\mathbf{B}(0,0)\big) - \mathbb{P}\big(\mathbf{A}(0,0) \cup \mathbf{B}(0,0)\big) \\ &\geq \mathbb{P}\big(\mathbf{A}(0,0)\big) + \mathbb{P}\big(\mathbf{B}(0,0)\big) - 1 \end{aligned} \tag{19}$$

Recall that

$$\mathbf{A}(0,0) = \big\{\mathbf{N}^{(y,k)}(m^*_{\lambda_0}) \leq (1 - \varepsilon_2) M_{\lambda_0}, \forall (y, k) \in \mathbf{X}\big\}$$

so if $\Delta := \text{Card}(\mathbf{X})$, we have, keeping in mind the mutual independence of the Poisson processes, that

$$\mathbb{P}\big(\mathbf{A}(0,0)\big) = (1 - a(\lambda_0))^{\Delta} \tag{20}$$

where

$$a(\lambda_0) = \mathbb{P}\big(\mathbf{N}(m^*_{\lambda_0}) > (1-\varepsilon_2)M_{\lambda_0}\big) = \mathbb{P}\bigg(\frac{\mathbf{N}(m^*_{\lambda_0})}{m^*_{\lambda_0}} - 1 > \frac{(1-\varepsilon_2)M_{\lambda_0}}{m^*_{\lambda_0}} - 1\bigg) \tag{21}$$

In the proof of Lemma 8., we will care to chose ε_2 such that we also have

$$m^*_{\lambda_0} \leq (1-2\varepsilon_2)M_{\lambda_0} \Longleftrightarrow M_{\lambda_0} \geq \frac{m^*_{\lambda_0}}{(1-2\varepsilon_2)} \tag{22}$$

Now,

$$\frac{(1-\varepsilon_2)M_{\lambda_0}}{m^*_{\lambda_0}} - 1 \geq \frac{(1-\varepsilon_2)}{m^*_{\lambda_0}} \frac{m^*_{\lambda_0}}{(1-2\varepsilon_2)} - 1 = \frac{1-\varepsilon_2-1+2\varepsilon_2}{1-2\varepsilon_2} = \frac{\varepsilon_2}{1-2\varepsilon_2} > \varepsilon_2 \tag{23}$$

and consequently

$$a(\lambda_0) \leq \mathbb{P}\bigg(\frac{\mathbf{N}(m^*_{\lambda_0})}{m^*_{\lambda_0}} - 1 > \varepsilon_2\bigg) \leq \exp\big(-C_1\varepsilon_2 m^*_{\lambda_0}\big) = \exp\bigg(-\frac{\widetilde{C_1}\varepsilon_2}{\lambda_0}\bigg) \tag{24}$$

where the second inequality comes from a standard large deviation result for the Poisson process (can be seen as a corollary to Theorem 16.).

Finally, taking a small enough λ_0 leads to small $a(\lambda_0)$, which translates to large $\mathbb{P}(\mathbf{A}(0,0))$.

Similarly, for

$$\mathbf{B}(0,0) = \bigg\{\sup_{t\geq K}\bigg|\frac{\mathbf{N}^{(y,k)}(t)}{t} - 1\bigg| \leq \delta, \forall (y,k) \in \mathbf{X}\bigg\} \tag{25}$$

we have that

$$\mathbb{P}\big(\mathbf{B}(0,0)\big) = (1 - b(K))^{\Delta} \tag{26}$$

where

$$b(K) = \mathbb{P}\bigg(\sup_{t\geq K}\bigg|\frac{\mathbf{N}^{(y,k)}(t)}{t} - 1\bigg| > \delta\bigg) \tag{27}$$

By the large deviation theorem (Theorem 16.) proven in the corresponding section in the Appendix, we also have that

$$b(K) \leq \exp(-\widetilde{C_2}\delta K) \tag{28}$$

so, making $b(K)$ small corresponds to taking a large K which, taking into account the proof of Lemma 8., again boils down to taking a small λ_0.

3.4 Proof of the deterministic lemma

Proof : We will first prove the lemma in the case $\kappa = 0$ when there is no interaction among the particles at different sites. Let K and δ be given and suppose that $\zeta_0(0)$ is $(\varepsilon_1, \varepsilon_2)$–occupied (the necessary ε_1 and ε_2 will be chosen at a later stage, but we may suppose that they are fixed to these chosen values at this point in the proof). We have to show that $\zeta_{n^*}(x)$ is also occupied for all $\|x\|_\infty \leq 1$, i.e. $\zeta_{n^*}(x) \in [\varepsilon_1 M_{\lambda_0}, (1-\varepsilon_2)M_{\lambda_0}]$ for all $\|x\|_\infty \leq 1$.

The upper bound follows easily by the $(B1)_{\varepsilon_2}$ hypothesis since we have

$$\zeta_{n+1}(x) = F(x, \zeta_n) + \delta_n(x) \leq (1-\varepsilon_2)M_{\lambda_0}$$

for all $(x, n) \in \mathbf{X}$ and in particular $\zeta_{n^*}(x) \leq (1-\varepsilon_2)M_{\lambda_0}$ for $\|x\|_\infty \leq 1 + R_\lambda$.

We establish the lower bound by induction, by showing that

$$\zeta_n(x) \geq p_{0x}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0} \tag{29}$$

for $0 \leq n \leq n^*$ and those x for which $(x, n) \in \mathbf{J}$. The fact that $\zeta_{n^*}(x) \geq \varepsilon_1 M_{\lambda_0}$ for all $\|x\|_\infty \leq 1$ will then follow by the definition of n^*.

The main idea is to adjust the parameters at our disposal — $\lambda_0^*, \varepsilon_1$ and ε_2 such that a large number of particles per site is maintained through all the epochs and such that the number of particles at every epoch and on every non–empty site is multiplied by at least $\widetilde{m}$. For $n = 0$, the claim follows by the assumption that $\zeta_0(0)$ is $(\varepsilon_1, \varepsilon_2)$–occupied. Suppose now that it holds for some $n < n^*$.

Let's assume for the moment that $F(\zeta_n, x) \geq K$ for every x such that $(x, n) \in \mathbf{J}$ so that we could use $(B2)_{\delta,K}$ and write

$$\zeta_{n+1}(x) = F(\zeta_n, x) + \delta_n(x) \geq (1-\delta)F(\zeta_n, x) \tag{30}$$

In the sequel, we will show, by appropriately choosing the parameters ε_1 and ε_2, that $(1-\delta)F(\zeta_n, x) \geq p_{0x}^{n+1}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0}$. Then, since

$$p_{0x}^{n+1}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0} = \frac{p_{0x}^{n+1}\widetilde{m}^{n+1}\varepsilon_1 m}{\lambda_0} \tag{31}$$

and the number of pairs (x, n) such that $p_{0x}^{n+1}\widetilde{m}^{n+1}$ is non–zero (namely, those are the pairs $(x, n) \in \mathbf{J}$), we can choose a sufficiently small λ_0^* such that for every $\lambda_0 \leq \lambda_0^*$ we have

$$p_{0x}^{n+1}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0} \geq K \tag{32}$$

for every $(x, n) \in \mathbf{J}$. In particular, $F(\zeta_n, x)$ would be greater than K for every $(x, n) \in \mathbf{J}$ and the inequality in (30) will be justified.

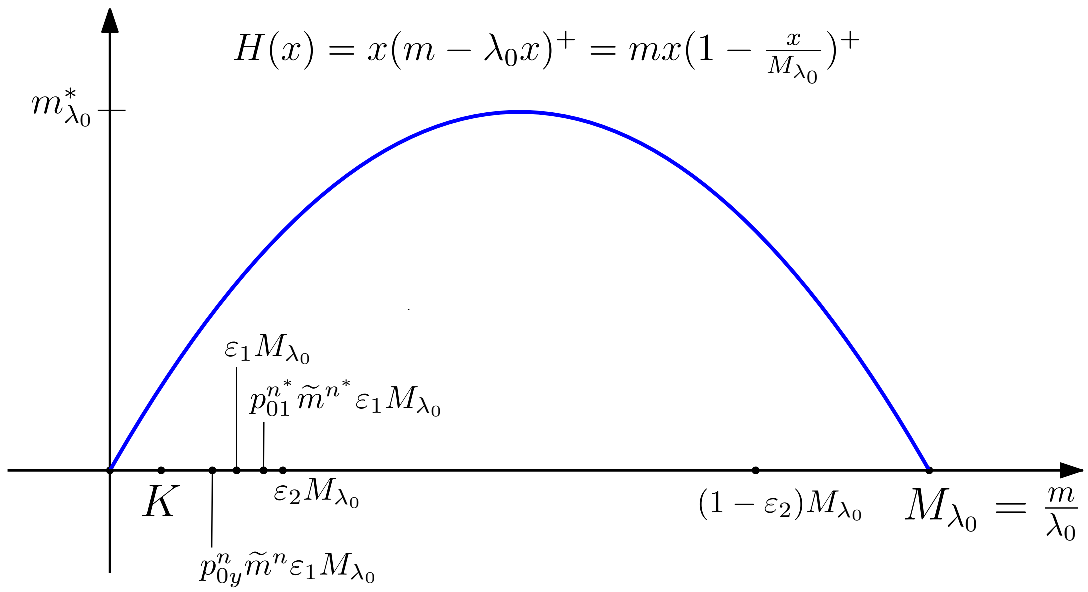

Figure 9: *Sketch of the function H. By choosing ε_1 such that $p_{0,x}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0}$ is on the left of $\varepsilon_2 M_{\lambda_0}$ we can claim that $H(\zeta_n(x)) \geq H(p_{0,x}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0})$ because $\zeta_n(x) \in [p_{0,x}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0}, (1-\varepsilon_2)M_{\lambda_0}]$.*

By the definition of $F(\zeta_n, x)$,

$$(1-\delta)F(\zeta_n, x) = (1-\delta) \sum_{y \in \mathbb{Z}^d} f(\zeta_n, y) p_{0y} = \sum_{y \in \mathbb{Z}^d} (1-\delta) f(\zeta_n, y) p_{0y} \tag{33}$$

so, it suffices to show that $(1-\delta)f(\zeta_n, y) \geq p_{0,y}^n \widetilde{m}^{n+1} \varepsilon_1 M_{\lambda_0}$. The Chapman–Kolmogorov relation will then give us the desired inequality.

By definition, $f(\zeta_n, y) = H(\zeta_n(y))$ and we know that $\zeta_n(y) \geq p_{0,y}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0}$ by the induction hypothesis. If we choose ε_1 such that $p_{0,y}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0} \leq \varepsilon_2 M_{\lambda_0}$ for all $(y, n) \in \mathbf{J}$, then, since $\zeta_n(x) \in [p_{0,x}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0}, (1-\varepsilon_2)M_{\lambda_0}]$, we could guarantee that

$$f(\zeta_n, y) = H(\zeta_n(y)) \geq H(p_{0,y}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0}) = p_{0,y}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0} m(1 - p_{0,y}^n \widetilde{m}^n \varepsilon_1) \tag{34}$$

and since $p_{0,y}^n \widetilde{m}^n \varepsilon_1 \leq \varepsilon_2$, we have

$$(1-\delta)f(\zeta_n, y) \geq (1-\delta)p_{0,y}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0} m(1-\varepsilon_2) = p_{0,y}^n \widetilde{m}^n \varepsilon_1 M_{\lambda_0} \times (1-\delta)m(1-\varepsilon_2) \tag{35}$$

so, at the end, it remains to choose ε_2 such that $(1-\delta)m(1-\varepsilon_2) \geq \widetilde{m}$. Notice that such a choice is possible because we are claiming the validity of the lemma for those δ's such that $(1-\delta)m > \widetilde{m}$.

In short, the steps of the proof are as follows and in the following order :

1. We choose ε_2 such that $(1-\delta)m(1-\varepsilon_2) \geq \widetilde{m}$.

1.b For reasons that become apparent in the proof of the main theorem, we also care to choose ε_2 at this point in such a way that we also have $m^*_{\lambda_0} \leq (1-2\varepsilon_2)M_{\lambda_0} \iff m^2/4\lambda_0 \leq (1-2\varepsilon_2)m/\lambda_0 \iff m \leq 4(1-2\varepsilon_2)$. Notice that the latter choice is possible because $m \in (1,4)$.

2. Then we choose ε_1 such that $p^n_{0,y}\widetilde{m}^n\varepsilon_1 \leq \varepsilon_2$ for every $(y,n) \in \mathbf{J}$. This choice implies that $p^n_{0,y}\widetilde{m}^n\varepsilon_1 M_{\lambda_0} \leq \varepsilon_2 M_{\lambda_0}$ for every $(y,n) \in \mathbf{J}$ and allows us to conclude that $(1-\delta)f(\zeta_n,y) \geq (1-\delta)H(p^n_{0,y}\widetilde{m}^n\varepsilon_1 M_{\lambda_0}) \geq p^n_{0,y}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0}$ and consequently, $(1-\delta)F(\zeta_n,x) \geq p^{n+1}_{0x}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0}$ for every $(x,n) \in \mathbf{J}$.

3. Finally, we make the appropriate choice of λ^*_0 which makes $F(\zeta_n,x) \geq K$ for all $(x,n) \in \mathbf{J}$, which, in turn, allows us to rely on $(B2)_{\delta,K}$ to conclude that $\zeta_{n+1}(x) \geq p^{n+1}_{0x}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0}$ and terminate the induction.

To handle the case of a non–zero neighborhood competition, in the first step above, we additionally choose a κ^* such that $(1-\delta)m(1-\varepsilon_2)(1-\kappa^*) \geq \widetilde{m}$.

Now, since

$$f(\zeta_n,y) = \zeta_n(y)\left(m - \lambda_0\zeta_n(y) - \sum_{z\neq y}\lambda_{yz}\zeta_n(z)\right)^+$$

and $\sum_{z\neq y}\lambda_{yz}\zeta_n(z) \leq (1-\varepsilon_2)M_{\lambda_0}\sum_{z\neq y}\lambda_{yz} = \kappa(1-\varepsilon_2)M_{\lambda_0}$, if $\kappa \leq \kappa^*\lambda_0$, we have that

$$\sum_{z\neq y}\lambda_{yz}\zeta_n(z) \leq \kappa^*\lambda_0(1-\varepsilon_2)M_{\lambda_0} = \kappa^*\lambda_0(1-\varepsilon_2)\frac{m}{\lambda_0} = \kappa^*(1-\varepsilon_2)m$$

and consequently,

$$f(\zeta_n,y) \geq \zeta_n(y)\Big(m - \lambda_0\zeta_n(y) - \kappa^*(1-\varepsilon_2)m\Big)^+ = \zeta_n(y)m\Big(1 - \zeta_n(y)/M_{\lambda_0} - \kappa^*(1-\varepsilon_2)\Big)^+.$$

In the same fashion as in the case $\kappa = 0$, we obtain that

$$(1-\delta)f(\zeta_n,y) \geq (1-\delta)p^n_{0,y}\widetilde{m}^n\varepsilon_1 M_{\lambda_0}m(1-\varepsilon_2-\kappa^*(1-\varepsilon_2)) =$$

$$p^n_{0,y}\widetilde{m}^n\varepsilon_1 M_{\lambda_0} \times (1-\delta)m(1-\varepsilon_2)(1-\kappa^*) \geq p^n_{0,y}\widetilde{m}^{n+1}\varepsilon_1 M_{\lambda_0}$$

and the rest of the proof proceeds exactly as before. The role of the small κ^* and the fact that we require κ itself to be smaller than $\kappa^*\lambda_0$ is only to ensure that the reproduction rate, albeit not m in the presence of competition, still remains above $\widetilde{m}$. □

The above lemma and the assumptions $(B1)_{\varepsilon_2}$ and $(B2)_{\delta,K}$ have been stated for the site $(0,0)$, but they can easily be adapted to any site $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+$ by a simple translation argument.

We now summarize the necessary steps to cook up a forever–living population and show how Theorem 7. is "applied" :

1. Take a site which has non–zero population at time $n = 0$, let it be x. $\xi_0(x)$ is the number of particles at x at time $n = 0$. Let $S = \max\{\xi_0(y) : \|y - x\|_\infty \leq R_\lambda\}$ be the maximum number of particles in the R_λ–neighborhood of x which will be our occupied site at the beginning.

2. We want to plant an infinite cluster of good sites at $(x, 0)$ to ensure that its occupancy will spread indefinitely in time. Our previous discussion tells us that such a percolating cluster will appear at $(x, 0)$ with positive probability if the probability of a site being fecund is larger than some p_1 given by Liggett–Schonmann–Stacey's theorem. We try to achieve $\mathbb{P}(\mathbf{X}(0, 0) \text{ is good}) \geq \mathbb{P}(\mathbf{A}(0, 0)) + \mathbb{P}(\mathbf{B}(0, 0)) - 1 \geq p_1$.

3. Fix an $\widetilde{m} \in (1, m)$ and then a $\delta > 0$ such that $m(1 - \delta) \geq \widetilde{m}$.

4. Choose a sufficiently large K such that $\mathbb{P}(\mathbf{B}(0, 0)) = (1 - b(K))^\Delta \geq \frac{1+p_1}{2}$, say.

5. For these values of δ and K, let Lemma 8. output the necessary ε_1, ε_2 and λ_0^* and intervene if necessary in the way in order to make x comply with our definition of an occupied site, that is :

- when choosing ε_2, take care to make it a bit smaller to also have $S \leq (1 - \varepsilon_2) M_{\lambda_0}$ such that x complies with (2) of the definition of an occupied site.
- when choosing ε_1, take care to make it small enough to also have $\xi_0(x) \geq \varepsilon_1 M_{\lambda_0}$ such that x satisfies (1) of the definition of an occupied site.

6. If the λ_0^* output by the lemma is not small enough to also have $\mathbb{P}(\mathbf{A}(0, 0)) = (1 - a(\lambda_0))^\Delta \geq \frac{1+p_1}{2}$, make it even smaller.

7. We are done ! Our site x is $(\varepsilon_1, \varepsilon_2)$-occupied at time $n = 0$ and an infinite cluster of good sites will appear there with non–zero probability because $\mathbb{P}(\mathbf{X}(0, 0) \text{ is good}) \geq p_1$.

3.5 Speed of the population propagation in space

Let's suppose that the Universe has chosen the right $\omega \in \Omega$ and the population does survive. A further question we might ask is how does it propagate into space ? If a certain number of individuals are present at the origin at the beginning of time, how will their descendants evolve and what portion of space will be inhabited in the long run ?

In Figure 10., we present a population evolving on $\mathbb{Z}$ at various times till $n = 160$ which started off with 10 particles at the origin and whose transition and competition kernel are symmetric. One thing we can observe is that the number of particles on the inhabited sites varies around the theoretical fixed point for the given parameters. Another thing is that the particles have spread up until certain point in space — at every epoch, the offspring pushes forward the boundary of the inhabited area creating a front that propagates into space.

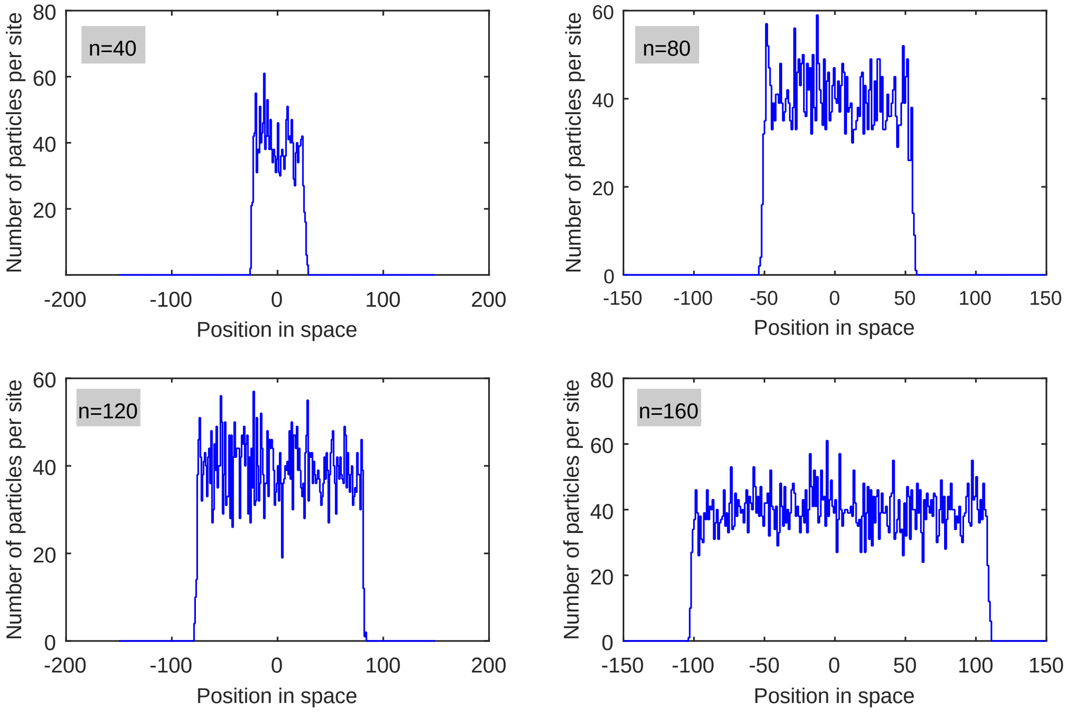

Figure 10: *MATLAB simulation depicting the population after 40, 80, 120 and 160 epochs. At time* $n = 0$*, there were only* 10 *particles at the origin. The parameters for this simulation were as follows :* $m = 2$*,* $p_{0,-1} = p_{0,1} = 0.25$*,* $p_{0,0} = 0.5$*,* $\lambda_{0,-1} = \lambda_{0,1} = 0.01$*,* $\lambda_0 = 0.005$*. The fixed point* $(m-1)/\sum_x \lambda_{0x} = 40$*. The right front is equal to* $R(160) = 111$*.*

To goal of this section will be to describe the way in which this expansion occurs for a population evolving in $\mathbb{Z}$. Namely, we will prove that the boundary of the inhabited area grows linearly with time by making use of existing results about the speed of branching random walks, a framework to which our model can easily be adapted.

Let $N(k)$ be the number of living particles at time k and $S_1(k), S_2(k), ..., S_{N(k)}(k)$ their positions at time k. We will be analyzing the propagation of the **right front** of the mass of particles. Clearly, it will be carried by the rightmost particle whose position we denote by $R(k)$. Note that there may be more than one particles at position $R(k)$ if they happen to share the same site. More precisely, $R(k)$ is defined as follows :

$$R(k) = \max_{1 \le i \le N(k)} S_i(k)$$

At first, we will suppose that there is no competition among particles (not even among those living on the same site) and that there is only one particle at the origin at time

0 ($\xi_0(0) = 1$, $\xi_0(x) = 0$ for $x \neq 0$). A theorem due to Biggins [9] will allow us to prove that the limit

$$\alpha = \lim_{k\to\infty} \frac{R(k)}{k} \tag{36}$$

exists almost surely on the event of non–extinction, i.e. that the right front evolves linearly as αk in the long run.

Define the function $\varphi(\theta)$ as the logarithm of the sum of the moment generating functions of the particles' positions at the first generation :

$$\varphi(\theta) = \log \mathbf{E}\Big[\sum_{i=1}^{N(1)} e^{\theta S_i(1)}\Big]. \tag{37}$$

We require that $\varphi(\theta) < \infty$ for every $\theta \in \mathbb{R}$, a condition which is satisfied in our case because the transition kernel $\mathbf{p}$ is finite–ranged (see the paragraph after the following lemma).

■ **Lemma 9.** $\varphi(\theta) = \log m + \log \mathbf{E}\big[e^{\theta S_1(1)}\big]$.

Proof.

$$\begin{aligned}
\varphi(\theta) &= \log \mathbf{E}\Big[\sum_{i=1}^{N(1)} e^{\theta S_i(1)}\Big] \\
&= \log \mathbf{E}\Big[\Big(\sum_{i=1}^{N(1)} e^{\theta S_i(1)}\Big) \sum_{k=1}^{\infty} \mathbb{1}[N(1) = k]\Big] \\
&= \log \sum_{k=1}^{\infty} \mathbf{E}\Big[\Big(\sum_{i=1}^{N(1)} e^{\theta S_i(1)}\Big) \mathbb{1}[N(1) = k]\Big] \\
&= \log \sum_{k=1}^{\infty} \mathbf{E}\Big[\sum_{i=1}^{k} e^{\theta S_i(1)}\Big] \mathbb{P}(N(1) = k) \\
&= \log \sum_{k=1}^{\infty} k\mathbb{P}(N(1) = k)\mathbf{E}\big[e^{\theta S_1(1)}\big] \\
&= \log \mathbf{E}[N(1)] + \log \mathbf{E}\big[e^{\theta S_1(1)}\big]
\end{aligned} \tag{38}$$

It remains to show that the expected number of particles at time $n = 1$ is m times the (expected) number of particles at time $n = 0$ — which is 1 because we supposed that $\xi_0(0) = 1$ and $\xi_0(x) = 0$ for all $x \neq 0$. This will follow once we argue that $\mathbf{E}[N(k)] = m\mathbf{E}[N(k-1)]$ for every k.

The number of particles at time k is a sum of **independent** Poisson random variables with means $F(\xi_{k-1}, y)$ where y ranges in $\mathbb{Z}$. As a consequence, $N(k)$ is again a Poisson random variable with mean

$$\sum_{y\in\mathbb{Z}} F(\xi_{k-1}, y) = \sum_{y\in\mathbb{Z}} \sum_{z\in\mathbb{Z}} \xi_{k-1}(z) m p_{z,y} = \sum_{z\in\mathbb{Z}} \xi_{k-1}(z) m = mN(k-1). \tag{39}$$

□

Now, it is easily seen that $\varphi(\theta) < \infty$ for every $\theta \in \mathbb{R}$ because

$$\mathbf{E}\big[e^{\theta S_1(1)}\big] = \int_{\mathbb{R}} e^{\theta x}\mathbf{p}(0,dx) = \sum_{x\in\mathbb{Z},|x|\le R_p} e^{\theta x} p_{0,x} < \infty. \tag{40}$$

Figure 11. below shows an accurate sketch of the function $\varphi(\theta)$ when the transition kernel $\mathbf{p} = [1/4, 1/2, 1/4]$. In that case, $\varphi(\theta) = \log m + \log(e^{-\theta}/4 + 1/2 + e^{\theta}/4) = \log m + \log(1/2 + \cosh(\theta)/2)$.

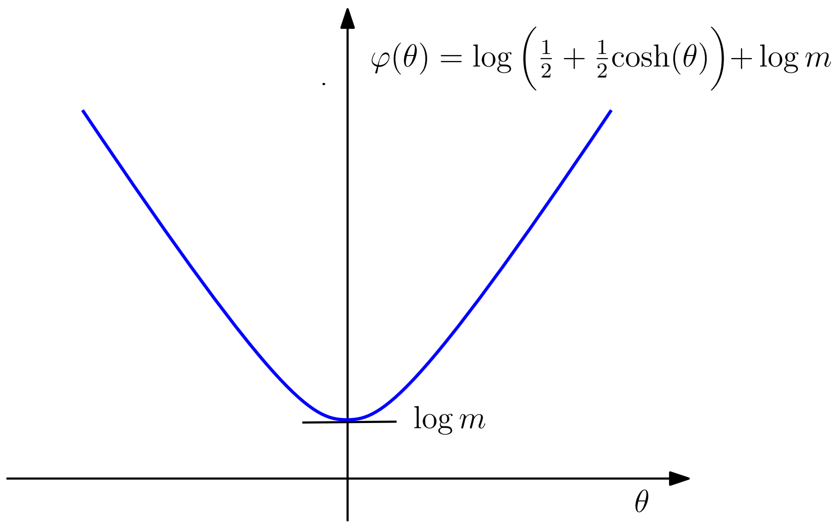

Figure 11: *Sketch of the function $\varphi(\theta)$ when the transition kernel $\mathbf{p}$ is given by $p_{0,1} = p_{0,-1} = 1/4$ and $p_{0,0} = 1/2$.*

We also define the function $\varphi^*(a)$ as the Legendre transform of the function $\varphi(\theta)$:

$$\varphi^*(a) = \sup_{\theta\in\mathbb{R}} \big[a\theta - \varphi(\theta)\big]. \tag{41}$$

The function $\varphi^*(a)$ plays a special role in that the speed of the right front $R(k)$ is given by

$$\alpha = \inf\big\{a \ge 0 : \varphi^*(a) > 0\big\}. \tag{42}$$

More precisely, we have the following theorem due to Biggins [9] :

■ **Theorem 10.** *Let $R(k) = \max_{1\le i\le N(k)} S_i(k)$ and $\alpha = \inf\big\{a \ge 0 : \varphi^*(a) > 0\big\}$ be as defined above. Then, $R(k)/k \to \alpha$ almost surely on the event of non–extinction.*

Clearly, once we introduce back the competition, the speed of the right front will decrease. Our aim now is to prove that the dependence of the speed on the competition is continuous, i.e. that any speed $\beta < \alpha$ might be achieved if one sufficiently reduces the competition.

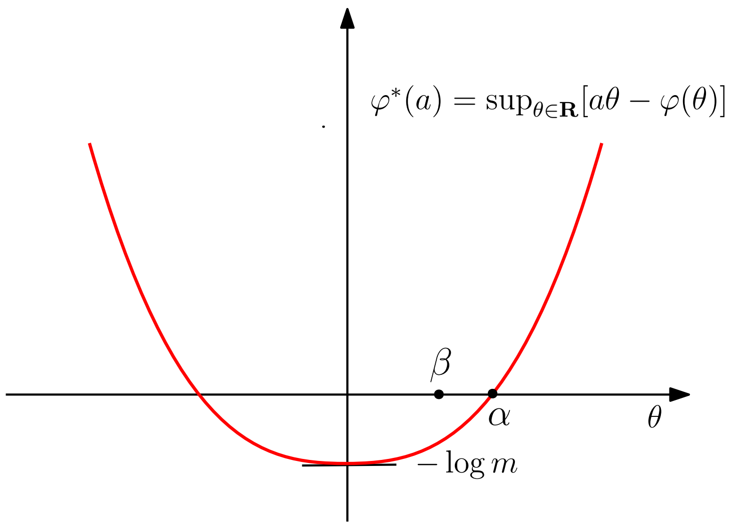

Figure 12: *Sketch of the function* $\varphi^*(a)$

We keep the definitions from the beginning of the section — the reproduction rate m, the transition kernel $\mathbf{p}$ which is supposed to satisfy (A1) and the competition kernel $\boldsymbol{\lambda}$ which is supposed to satisfy (A2). In the sequel we will assume they are defined and known.

The idea is to follow the example of the proof of Theorem 7. More precisely, we will prove an analogue of Lemma 8. (with another n^* suitably chosen) which says that an occupied site at time $n = 0$ remains occupied at time $n = n^*$ and makes the site on position $\lceil \beta n^* \rceil$ on the right occupied as well. This will lead us to consider yet another model of oriented percolation (which we will call the **new** model) where the edges are between sites

$$(kn^*, l\lceil \beta n^* \rceil) \to ((k+1)n^*, l\lceil \beta n^* \rceil)$$

and

$$(kn^*, l\lceil \beta n^* \rceil) \to ((k+1)n^*, (l+1)\lceil \beta n^* \rceil)$$

for $k \geq 0$ and $0 \leq l \leq k$. See Figure 13. for an illustration with $n^* = 3$ and $\beta = 1$. This new model is essentially a skewed version of Durrett's percolation model with only the origin present at the beginning.

Now, if the percolating cluster in the new model could extend arbitrarily close to the right border for a sufficiently high percolation parameter (i.e. sufficiently small competition), it would imply that the speed of the right front $R(k)$ could be made arbitrarily close to β.

■ **Lemma 11.** *Let α be the speed of propagation of the right front of the population in the absence of competition and let $\beta < \alpha$. Define $n^* = n^*(\mathbf{p}, \beta, \widetilde{m})$ by*

$$n^* = \min\left\{ n \in \mathbb{N} : p_{0,x}^n \widetilde{m}^n \geq 1 \text{ for } x = 0 \text{ and } x = \lceil \beta n \rceil \right\}.$$

There exists $\widetilde{m} < m$ such that n^* is finite.

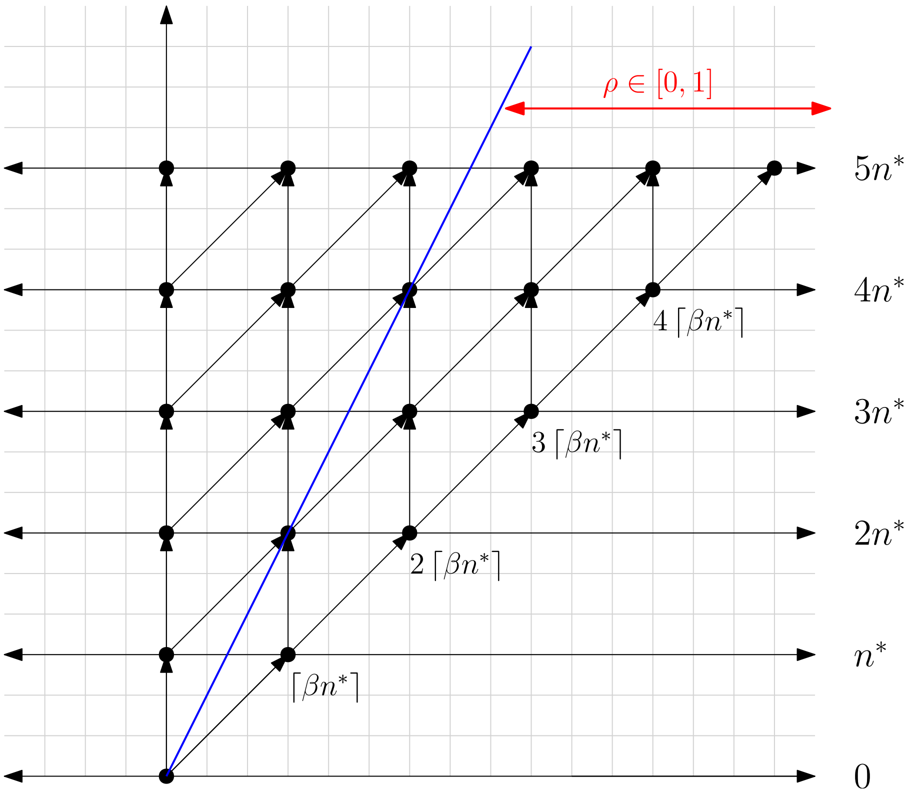

Figure 13: *The sketch of the new percolation grid if $n^* = 3$ and $\beta = 1$: a skewed version of Durrett's percolation model. A speed of ρ for its infinite cluster right end translates to a speed of $(1+\rho)\beta/2$ for the right front of the spatial branching system*

Proof. The main problem is to prove that $p^n_{0,\lceil \beta n \rceil}\widetilde{m}^n \geq 1$ for n sufficiently large. The fact that $p^n_{0,0}\widetilde{m}^n \geq 1$ for some finite n follows from the standard local central limit theorem as before. We will show that $p^n_{0,\lceil \beta n \rceil}\widetilde{m}^n$ tends to infinity as n gets larger by essentially following the usual proof of Cramér's theorem using a local central limit theorem instead of the standard central limit theorem to conclude.

Let $M(\theta)$ be the moment generating function of the transition kernel $\mathbf{p}$, i.e. $M(\theta) = \int e^{\theta x}\mathbf{p}(0,dx)$, $\Lambda(\theta) = \log M(\theta)$ the corresponding log–moment generating function and let the associated rate function be $I(b) = \sup_{\theta\in\mathbb{R}}\{b\theta - \Lambda(\theta)\}$. Suppose that the supremum for $I(\beta)$ $(b = \beta)$ is achieved at a point $\theta^* \in \mathbb{R}$, i.e. $I(\beta) = \beta\theta^* - \Lambda(\theta^*)$.

The idea of the proof is to relate the behavior of $p^n_{0,\lceil \beta n \rceil}$ to the same quantity of a modified transition kernel, $\widetilde{\mathbf{p}}$, whose mean will be β. More precisely, we define

$$\widetilde{p}_{0,x} = p_{0,x}\exp[x\theta^* - \Lambda(\theta^*)] \tag{43}$$

for every $x \in \mathbb{Z}$. To see that the mean of $\widetilde{\mathbf{p}}$ is β, notice that its generating function

$$\widetilde{M}(\theta) = \int e^{\theta x}\widetilde{\mathbf{p}}(0,dx) = \int e^{\theta x}e^{\theta^* x - \Lambda(\theta^*)}\mathbf{p}(0,dx) = M(\theta+\theta^*)e^{-\Lambda(\theta^*)}. \tag{44}$$

Now, the mean of $\widetilde{\mathbf{p}}$, which is given by

$$\frac{d\widetilde{M}(\theta)}{d\theta}\bigg|_{\theta=0} = M'(\theta^*)e^{-\Lambda(\theta^*)} = M'(\theta^*)/M(\theta^*) \tag{45}$$

is indeed equal to β because θ^* achieves the supremum for $I(\beta)$ and therefore

$$\frac{d}{d\theta}\big[\beta\theta - \log M(\theta)\big]\bigg|_{\theta=\theta^*} = 0 \Longleftrightarrow \beta - M'(\theta^*)/M(\theta^*) = 0. \tag{46}$$

The new kernel $\widetilde{\mathbf{p}}$ can be related to $\mathbf{p}$ via the following relation (valid for every n) :

$$\widetilde{p}^n_{0,x} = p^n_{0,x}\exp[x\theta^* - n\Lambda(\theta^*)] \Longleftrightarrow p^n_{0,x} = \widetilde{p}^n_{0,x}\exp[n\Lambda(\theta^*) - x\theta^*] \tag{47}$$

which can be proven by induction. Indeed, suppose that we have $\widetilde{p}^n_{0,x} = p^n_{0,x}\exp[x\theta^* - n\Lambda(\theta^*)]$. Then,

$$\begin{aligned}\widetilde{p}^{n+1}_{0,x} &= \sum_{y\in\mathbb{Z}} \widetilde{p}^n_{0,y}\widetilde{p}_{y,x} = \sum_{y\in\mathbb{Z}} p^n_{0,y}\exp[y\theta^* - n\Lambda(\theta^*)] \times p_{y,x}\exp[(x-y)\theta^* - \Lambda(\theta^*)] \\ &= \exp[x\theta^* - (n+1)\Lambda(\theta^*)]p^{n+1}_{0,x}\end{aligned} \tag{48}$$

where we used the fact that $p_{y,x} = p_{0,x-y}$.

Now, since $\Lambda(\theta^*) = \beta\theta^* - I(\beta)$, we have, using the second identity of equation (47),

$$p^n_{0,\lceil\beta n\rceil} = \widetilde{p}^n_{0,\lceil\beta n\rceil}\exp[n\Lambda(\theta^*) - \lceil\beta n\rceil\,\theta^*] = \widetilde{p}^n_{0,\lceil\beta n\rceil}\exp[-nI(\beta) + \beta n\theta^* - \lceil\beta n\rceil\,\theta^*] \tag{49}$$

Furthermore, $I(\beta) = \sup_{\theta\in\mathbb{R}}\{\beta\theta - \log\int e^{\theta x}\mathbf{p}(0,dx)\}$ and since

$$\log\int e^{\theta x}\mathbf{p}(0,dx) = \log\mathbf{E}[e^{\theta S_1(1)}] = \varphi(\theta) - \log m$$

by Lemma 9., we obtain that

$$I(\beta) = \log m + \sup_{\theta\in\mathbb{R}}\{\beta\theta - \varphi(\theta)\} = \log m + \varphi^*(\beta).$$

Finally, multiplying by $\widetilde{m}^n$,

$$p^n_{0,\lceil\beta n\rceil}\widetilde{m}^n = \widetilde{p}^n_{0,\lceil\beta n\rceil}\exp[-n(\varphi^*(\beta) + \log m - \log\widetilde{m}) + (\beta n - \lceil\beta n\rceil)\theta^*] \tag{50}$$

which tends to ∞ as $n \to \infty$. To see this, note that the quantity $\exp[(\beta n - \lceil\beta n\rceil)\theta^*]$ is bounded by a constant, the term $\widetilde{p}^n_{0,\lceil\beta n\rceil} \sim C/\sqrt{n}$ by the local central limit theorem for $\widetilde{\mathbf{p}}$ (see Theorem 15. in the Appendix) and it remains to show that $-\big(\varphi^*(\beta) + \log m - \log\widetilde{m}\big) > 0$, but once we have fixed β, we can choose an $\widetilde{m}$ sufficiently close to m to assure the inequality. See Figure 12. and notice that $\varphi^*(\beta) < 0$ by the choice of β.

□

Once we have established Lemma 11., the proof of the following lemma is analogous to the proof of the deterministic lemma of the preceding section.

■ **Lemma 12.** *Let α be the speed of propagation of the right front of the population in the absence of competition and let $\beta < \alpha$. For each $K > 0$ and δ satisfying $m(1-\delta) > \widetilde{m} > 1$, there are choices of positive numbers $\varepsilon_1, \varepsilon_2, \lambda_0^*$ and κ^* such that whenever $\lambda_0 \le \lambda_0^*$ and $\kappa \le \kappa^* \lambda_0$, the following holds :*

$$\zeta_0(0) \text{ is } (\varepsilon_1, \varepsilon_2)\text{-occupied}, (B1)_{\varepsilon_2}, (B2)_{\delta,K} \text{ are satisfied } \implies$$
$$\zeta_{n^*}(0) \textbf{ and } \zeta_{n^*}(\lceil \beta n^* \rceil) \text{ are } (\varepsilon_1, \varepsilon_2)\text{-occupied .}$$

In the same way as in Theorem 7., this lemma allows us to conclude that the population survives with positive probability (the good sites on the new grid percolate and give rise to an infinite chain of occupied sites). We show that by possibly reducing the competition even more, we can achieve the desired speed β for $R(k)$ on this event of survival. More precisely, we have the following theorem :

■ **Theorem 13.** *Let α be the speed of propagation of the right front of the population in the absence of competition and let $\beta < \alpha$. There are choices of positive numbers λ_0^* and κ^* such that if $\lambda_0 \le \lambda_0^*$ and $\sum_{x \ne 0} \lambda_{0x} \le \kappa^* \lambda_0$, then*

$$\liminf_{k\to\infty} \frac{R(k)}{k} \ge \beta$$

on a non–zero probability event on which the population survives.

As mentioned previously, there is a correspondence between Durrett's oriented percolation model and the new oriented percolation model we are considering in this section. The blue line in Figure 13. (the line that starts at the origin and passes through the middle site in every even epoch) corresponds to the y–axis in Durrett's model. We remind that the right end of the infinite cluster appearing at the origin is denoted by $r(k)$ and achieves a speed of $\rho \in (0, 1]$ almost surely on the event of percolation (Theorem 2.). Moreover, $\rho \to 1$ as the parameter $p \to 1$ (Theorem 3.).

The maximal span at time k of the infinite cluster arising at the origin in Durrett's model is $[-k, k]$, while in our model (at time kn^*) is $[0, k\lceil \beta n^* \rceil]$. By a simple observation, we notice that a speed of ρ for $r(k)$ corresponds to a speed of $\frac{(1+\rho)\beta}{2}$ for $R(k)$. In particular, $R(k) \to \beta$ as $p \to 1$.

To prove Theorem 13., we use Lemma 12. with $\beta' > \beta$ and proceed in the same fashion as the proof of Theorem 7., the only differences being the use of the new oriented percolation grid and the fact that we may need to reduce the competition even more to achieve the desired speed β of the right front of the percolating cluster.

4 Appendix

4.1 Local central limit theorem

4.1.1 Zero–mean transition kernel

In this section, we are going to establish the fact that the number n^* defined in section 3.3.1. is finite. We recall its definition :

$$n^* = \min\left\{n \in \mathbb{N} : p^n_{0,x}\widetilde{m}^n \geq 1 \quad \forall x \text{ such that } \|x\|_\infty \leq 1\right\} \tag{51}$$

where $p^n_{0,x}$ is the n–step transition probability from 0 to x and $\widetilde{m}$ is the guaranteed multiplication factor.

We will make use of a standard local central limit theorem for a random walk on the d–dimensional integer lattice which asserts that $p^n_{0,x}$ decays at a polynomial rate if certain conditions are satisfied by $\mathbf{p}$. More precisely, we have the following result :

■ **Theorem 14.** *Let* $\mathbf{p}$ *be an irreducible aperiodic transition kernel, defined by a measure* $\boldsymbol{\mu}$ *on* $\mathbb{Z}^d$ *(*$p_{x,y} = \boldsymbol{\mu}(y-x)$ *and thus* $p_{0,x} = \boldsymbol{\mu}(x)$*) which has zero–mean and finite variance. Then, for every* $x \in \mathbb{Z}^d$*, we have*

$$p^n_{0,x} = \boldsymbol{\mu}^{*n}(x) \sim \frac{C}{n^{d/2}} \tag{52}$$

where C *is a constant depending on* $\boldsymbol{\mu}$*.*

The transition kernel $\mathbf{p}$ from the spatial branching model satisfies the hypotheses of the previous theorem and, since the number of $x \in \mathbb{Z}^d$ such that $\|x\|_\infty \leq 1$ is finite, we deduce that n^* is finite as well.

Theorem 14. is obtained as a corollary to a more precise result which can be found in [7] (Chapter III. 13, Theorem (13.10) and Corollary (13.11)).

4.1.2 Non zero–mean transition kernel

A similar result holds in the case where the transition kernel has a drift. The theorem below is a paraphrase of a slightly more general result proved in [10] (Chapter VII. 1, Theorem 1.). We use it to prove that the redefined n^* in section 3.5. is finite.

■ **Theorem 15.** *Let* $\widetilde{\mathbf{p}}$ *be an irreducible aperiodic transition kernel taking values in* $\mathbb{Z}$ *with mean* β *and variance* $\sigma^2 > 0$*. Then,*

$$\sup_x \left|\sigma\sqrt{n} \times \widetilde{p}^n_{0,x} - \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-n\beta}{\sigma\sqrt{n}}\right)^2\right\}\right| \to 0. \tag{53}$$

By taking $x = \lceil \beta n \rceil$, we have that

$$\widetilde{p}^n_{0,\lceil \beta n \rceil} \sim \frac{C}{\sqrt{n}} \tag{54}$$

as $n \to \infty$.

4.2 Poission process large deviations

■ **Theorem 16.** *Let* $\mathbf{N}$ *be a standard Poisson process* ($\mathbf{N}(t)$ *is a Poisson random variable with mean* t*). Then, there is a constant* C *such that for every* $\delta > 0$*,*

$$\mathbb{P}\bigg(\sup_{t\geq K}\bigg|\frac{\mathbf{N}(t)}{t}-1\bigg|>\delta\bigg)\leq \exp\big(-C\delta K\big).$$

The proof of Theorem 16. will make use of the Doob's L^p martingale inequality (for $p = 1$) which we recall here without proof.

■ **Theorem 17** (Doob's L^1 martingale inequality)**.** *Let* $(X_t)_{t\geq 0}$ *be a non–negative continuous time càdlàg martingale. That is, for all times* s *and* t *with* $s < t$*,* $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$*. Then, for any constant* C*,*

$$\mathbb{P}\bigg(\sup_{0\geq t\geq T} X_t \geq C\bigg) \leq \frac{\mathbf{E}[X_T]}{C} \tag{55}$$

Proof. (of Theorem 16.) Let $\delta > 0$. We will show that

$$\mathbb{P}\bigg(\sup_{t\geq K}\frac{\mathbf{N}(t)}{t}-1\geq\delta\bigg)\leq \exp\big(-C'\delta K\big)$$

The (standard) Poisson process $\mathbf{N}(t)$ is in particular a Levy process with

$$\mathbf{E}[e^{\theta\mathbf{N}(t)}] = e^{t\psi(\theta)} \tag{56}$$

where $\psi(\theta)$ is the characteristic exponent of $\mathbf{N}(t)$, a random variable with infinitely divisible distribution. More precisely,

$$\psi(\theta) = \log\mathbf{E}[e^{\theta\mathbf{N}(1)}] = \log e^{(e^{\theta}-1)} = e^{\theta}-1. \tag{57}$$

Consequently, the process $M_t = e^{\theta\mathbf{N}(t)-t\psi(\theta)}$ is a non–negative martingale. We fix a time $T > K$

$$\begin{aligned}
\mathbb{P}\bigg(\sup_{T\geq t\geq K}\frac{\mathbf{N}(t)}{t}-1\geq\delta\bigg) &= \mathbb{P}\bigg(\exists t\in[K,T] \quad : \quad \frac{\mathbf{N}(t)}{t}\geq\delta+1\bigg)\\
&= \mathbb{P}\bigg(\exists t\in[K,T] \quad : \quad \mathbf{N}(t)\geq t(\delta+1)\bigg)\\
&= \mathbb{P}\bigg(\exists t\in[K,T] \quad : \quad \theta\mathbf{N}(t)\geq \theta t(\delta+1)\bigg)\\
&= \mathbb{P}\bigg(\exists t\in[K,T] \quad : \quad e^{\theta\mathbf{N}(t)-t\psi(\theta)}\geq e^{\theta t(\delta+1)-t\psi(\theta)}\bigg)\\
&= \mathbb{P}\bigg(\exists t\in[K,T] \quad : \quad e^{\theta\mathbf{N}(t)-t\psi(\theta)}\geq e^{t(\theta(\delta+1)-\psi(\theta))}\bigg)
\end{aligned} \tag{58}$$

Now, if $\theta(\delta+1)-\psi(\theta)\geq 0$, we have, using the Doob's martingale inequality, that

$$\begin{aligned}\mathbb{P}\left(\sup_{T\geq t\geq K}\frac{\mathbf{N}(t)}{t}-1\geq\delta\right)&\leq\mathbb{P}\left(\exists t\in[K,T]\quad:\quad e^{\theta\mathbf{N}(t)-t\psi(\theta)}\geq e^{K(\theta(\delta+1)-\psi(\theta))}\right)\\&\leq\frac{\mathbf{E}[e^{\theta\mathbf{N}_T-T\psi(\theta)}]}{e^{K(\theta(\delta+1)-\psi(\theta))}}=\frac{1}{e^{K(\theta(\delta+1)-\psi(\theta))}}\end{aligned}\tag{59}$$

By optimizing on θ, we have that

$$\mathbb{P}\left(\sup_{T\geq t\geq K}\frac{\mathbf{N}(t)}{t}-1\geq\delta\right)\leq\exp(-K\psi^*(\delta+1))\tag{60}$$

where $\psi^*(\delta+1)=\sup_\theta\{(\delta+1)\theta-\psi(\theta)\}$. To have a meaningful bound, we need $\psi^*(\delta+1)>0$ which is equivalent to $\delta+1>\psi'(0)$. Since $\psi'(0)=\mathbf{E}[\mathbf{N}(1)]=1$ and $\delta>0$, the last inequality is satisfied. Note that we also have $\theta(\delta+1)-\psi(\theta)\geq 0$ for the minimizing θ. □

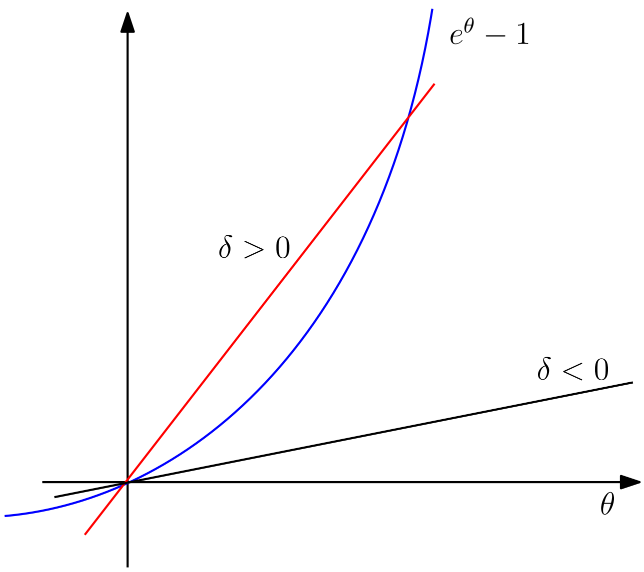

Figure 14: *Sketch of the characteristic exponent $\psi(\theta)$ of the standard Poisson process, along with two lines $l(\theta)=(\delta+1)\theta$ corresponding to $\delta>0$ and $\delta<0$. When $\delta>0$, all the reasoning in the proof is valid.*

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